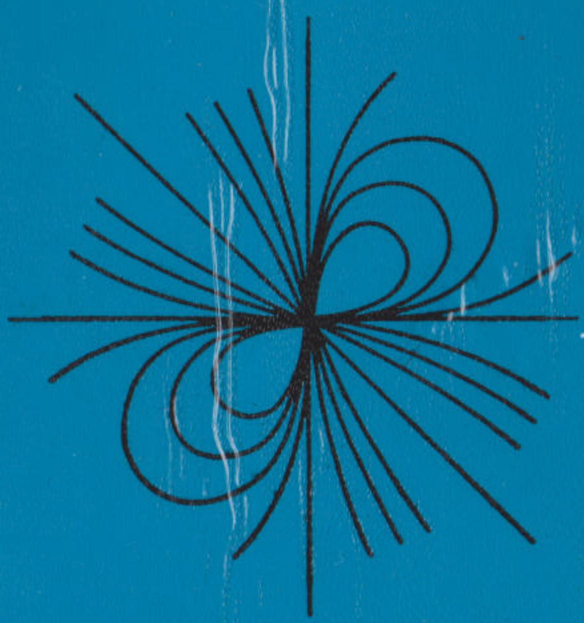
R. L. E. SCHWARZENBERGER

Elementary Differential Equations

CHAPMAN AND HALL



Elementary Differential Equations

CHAPMAN AND HALL MATHEMATICS SERIES

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Elementary Differential Equations

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CHAPMAN AND HALL LTD

11 NEW FETTER LANE LONDON EC4

First published 1969
Reprinted 1971

Printed in Great Britain by
Fletcher & Son Ltd,
Norwich
SBN 412 09580 7

Distributed in the U.S.A.
by Barnes & Noble, Inc.

Preface

The aim of this book is to give a rapid introduction to differential equations which is elementary and which emphasizes the geometric nature of the subject.

The book is elementary in the sense that it assumes only a knowledge of the simplest properties of differentiation and integration. It is designed primarily for first-year college and university students who have not yet had courses in analysis or linear algebra. For this reason topics which are of only specialized interest (such as approximation techniques and numerical methods of solution) and more advanced topics which require linear algebra and analysis (such as existence theorems, partial differential equations and series solutions) are either omitted or mentioned briefly in the guide to further reading. Occasionally, in order to give a precise statement of a definition or proof, it is necessary to use a property of continuous functions from an analysis course. The reader should accept such properties – which are usually 'intuitively' obvious – until he feels ready to look up further details in a book on analysis, such as *A Preliminary Course in Analysis* by R. M. F. Moss and G. T. Roberts (which has appeared in the same series, and is in the sequel referred to simply as Moss–Roberts). In this way the book should be suitable both for students specializing in mathematics and for students in other subjects who need differential equations as a tool. The latter group are warned, however, that they will not find the customary large number of examples of differential equations arising in biology, chemistry, economics, engineering and physics. The reason is that

such students are likely to study many examples in their own work, and are not likely to be helped by examples from a background which they do not understand.

The approach is geometric, as can be seen from the large number of diagrams. The main reason is that most mistakes in differential equations result from blindly manipulating formulae without thinking about the geometry involved. A geometric point of view is often necessary for applications, in which the required 'solution' of a differential equation is a picture rather than a formula. Finally, the geometric theory of differential equations has during the last hundred years developed from the elementary beginnings described in this book to an exciting area of research in which algebra, analysis and topology all play a vital role.

I am grateful to Ronald Brown for his suggestion that such a book should appear in the Chapman and Hall Mathematics Series, to Oxford Illustrators Ltd for their help with the diagrams, and to the publishers and printers for their careful and sympathetic co-operation.

R.L.E.S.

Contents

Glossary of Symbols

Symbol	*Meaning*	
cos, sin	cosine, sine	*page* 2
cosh, sinh	hyperbolic cosine, hyperbolic sine	33
exp	exponential	16
log	logarithm to base $e = \exp 1$	16
re, im	real part, imaginary part	58
$\mathbf{R}$	line of real numbers	7
$\mathbf{R}^2$	plane of pairs of real numbers	7
$\mathbf{V}$	set of infinitely differentiable functions f with fixed domain	72
$\mathbf{W}$	vector space	87
$\mathscr{D}$	domain of function	10
$\mathscr{S}$	region in $\mathbf{R}^2$	8
$\in$	belongs to	8
$f : \mathscr{D} \to \mathbf{R}$	function f which associates real number $f(t)$ to each $t \in \mathscr{D}$	89
$t \mapsto f(t)$	function f which associates value $f(t)$ to each t in domain	11
$f', Df, \dfrac{df}{dt}$	derivative of function	13
$\dfrac{\partial f}{\partial t}, \dfrac{\partial f}{\partial x}$	partial derivatives of $f(t, x)$ with respect to t, x	41
$\sqrt{x}$	positive square root (real number $y \geqslant 0$ such that $y^2 = x$) of real number $x \geqslant 0$	5

$\sqrt[3]{x}$	cube root (real number y such that $y^3 = x$) of real number x	27
$\lvert x \rvert$	modulus (x if $x \geqslant 0$ and $-x$ if $x \leqslant 0$) of real number x	30
$\{x : P(x)\}$	set of all x such that property $P(x)$ holds (often context implies $x \in \mathbf{R}$)	8

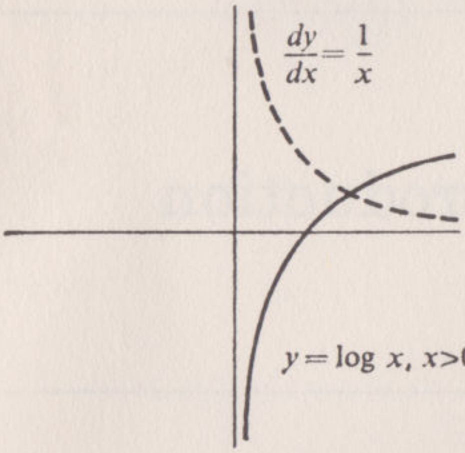
Introduction

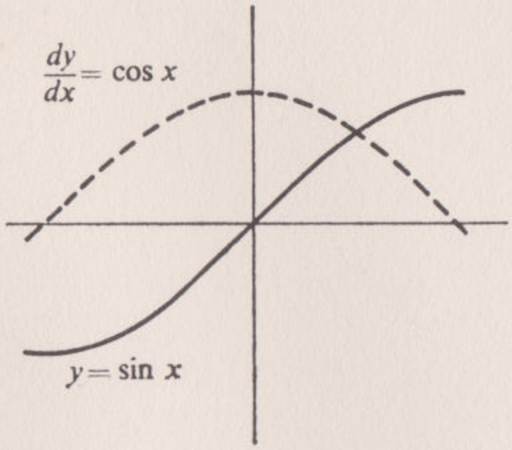
This book is based on a course of twenty lectures given to first-year mathematics students at the University of Warwick. One aim of the course was to provide an antidote to the 'abstract' courses on sets and groups which the students attended simultaneously in their first term. Another was to stimulate geometric intuition and to provide motivation for the courses on analysis and linear algebra which the students attended in the following term.

These aims implied already that the course should be 'concrete' rather than 'abstract'. There is, however, another reason for this approach. It is a common complaint among undergraduates that university mathematics seems unnecessarily abstract compared with the concrete mathematics they meet at school. Another aim of this book is therefore to deal with differential equations in a rather concrete manner, and to show how this leads naturally to such 'abstract' concepts as continuous function, vector space and linear operator.

In fact it is sometimes the school – rather than university – treatment of differential equations which is unnecessarily abstract. Consider for example the differentiation of functions like $y = \log x$ and $y = \sin x$. It is familiar that the derivatives are $1/x$ and $\cos x$. It may seem pedantic to point out that this is not quite accurate, since the function $y = \log x$ makes sense only for $x > 0$, whereas its alleged derivative $1/x$ makes sense both for $x > 0$ and for $x < 0$.

To see why such pedantry may be very necessary, consider the differentiation of a function of a function. The well-known chain





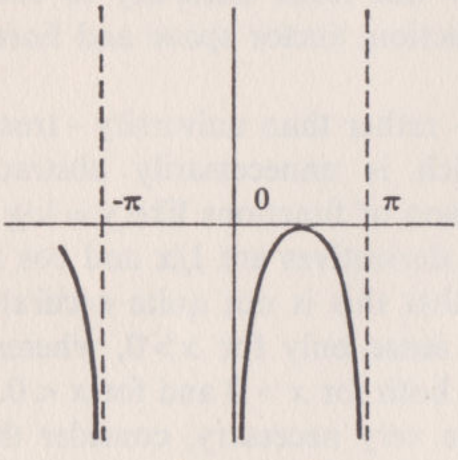
rule gives

$$\frac{d}{dx}(\log \sin x) = \frac{\cos x}{\sin x}$$

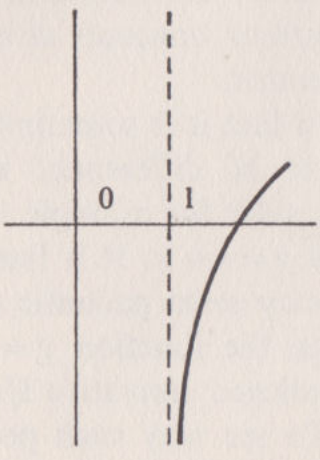
$$\frac{d}{dx}(\log \log x) = \frac{1}{x \log x}$$

$$\frac{d}{dx}(\log \log \sin x) = \frac{\cos x}{\sin x \log \sin x}.$$

However, these computations are somewhat abstract: an attempt to draw the graphs of the functions involved shows that the first two equations are often meaningless, while the third equation is always meaningless. Since $y = \log x$ makes sense only for $x > 0$, the function $y = \log \sin x$ makes sense only for points x lying strictly between an even multiple of π and the next odd multiple of π. On the other hand its alleged derivative makes sense for all points x (except for an infinite value at even multiples of π).



$y = \log \sin x$



$y = \log \log x$

Similarly $y = \log \log x$ makes sense only for $x > 1$ whereas its alleged derivative makes sense for any $x > 0$. Finally the function $y = \log \log \sin x$ has no graph at all because $\sin x \leqslant 1$ and therefore $\log \sin x \leqslant 0$. Whatever the formula involving $y = \log \log \sin x$ means, it is certainly not a piece of concrete mathematics.

It may seem artificial to consider such a bizarre function, but the same kind of alarming situation can arise from the simplest differential equations.

What is wrong with the following argument? 'Since

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

the solution of the equation $dy/dx = 1/x$ is $y = \log x$.'

There are two mistakes. Firstly inspection shows that equally well

$$\frac{d}{dx}(\log(-x)) = \frac{1}{x}$$

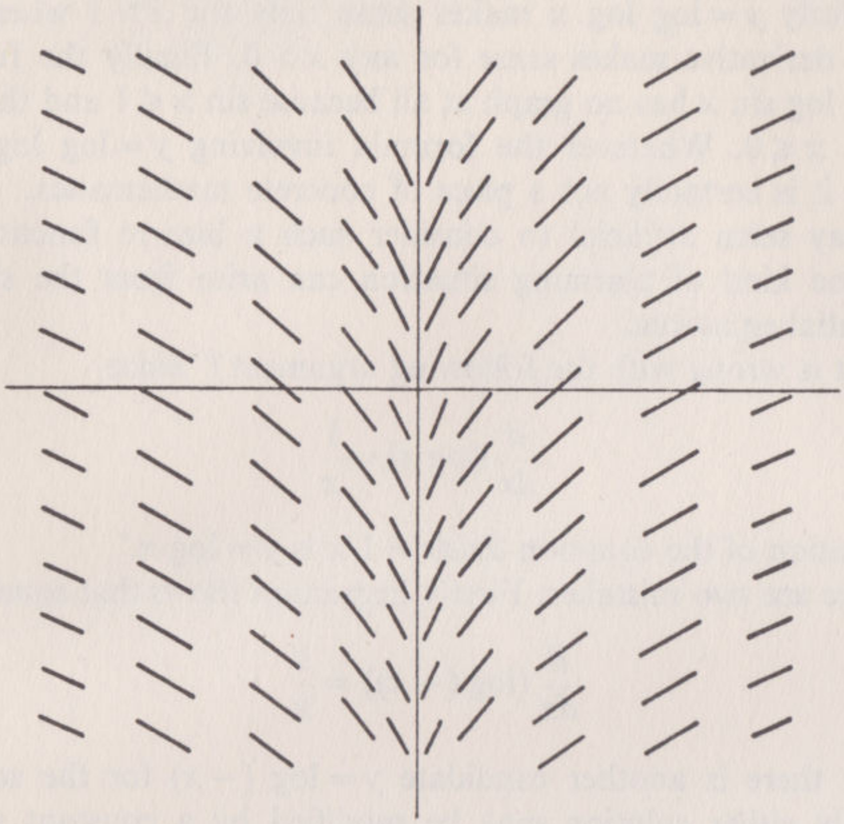
so that there is another candidate $y = \log(-x)$ for the solution. Secondly either solution may be modified by a constant without affecting the derivative. Therefore any of the functions

$$y = a + \log x$$

$$y = b + \log(-x)$$

(where a, b are real numbers) are solutions of the differential equation $dy/dx = 1/x$. This result clearly contradicts the statement, which is commonly made, that if $(d/dx)(f(x)) = (d/dx)(g(x))$ then there is a real number c such that $f(x) = g(x) + c$. The reason is that the statement is false unless more is said about the precise nature of the functions $y = f(x)$ and $y = g(x)$. It is not enough that they are represented by formulae; as we have seen in this area abstract formulae, however impressive, can be meaningless.

Let us now get away from abstract manipulations of formulae, and consider the equation $dy/dx = 1/x$ from a concrete point of view. The equation is an instruction to find functions $y = f(x)$ whose graphs have slope $1/x$ at the point $(x, f(x))$. It is necessary to exclude the point $x = 0$, since here $1/x$ is undefined and so the equation says nothing. At all other points the slope $1/x$ may be represented by a non-vertical direction. The diagram of slopes is symmetrical under reflection in the vertical axis. It is clear intuitively that the

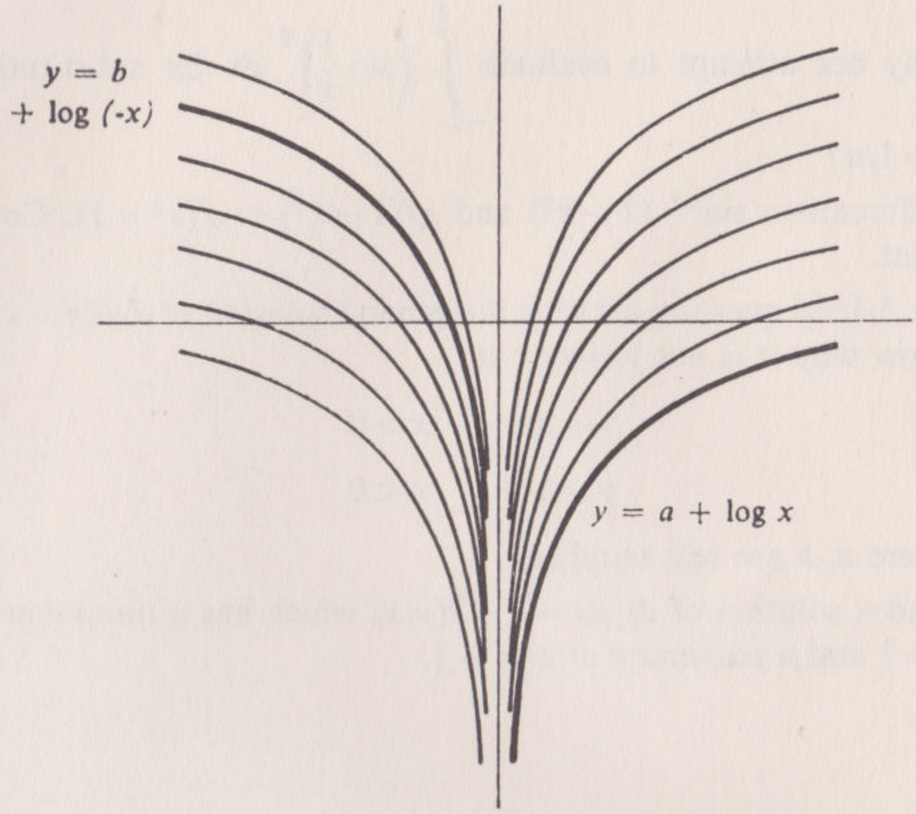


region $x > 0$ can be filled by curves with the required slopes; these are in fact the graphs of the functions $y = a + \log x$. A check shows that exactly one such graph passes through each point (x, y) with $x > 0$. Similarly the curves filling the left-hand region $x < 0$ are the graphs of the functions $y = b + \log(-x)$. Therefore the correct solution to the equation $dy/dx = 1/x$ is

$$y = a + \log x, \quad x > 0$$

$$y = b + \log(-x), \quad x < 0$$

where a, b are real numbers (sometimes called *arbitrary constants*; we prefer to avoid the use of this phrase because it is commonly asserted that if a differential equation involves dy/dx but no higher derivatives then its solution must contain only *one* arbitrary constant – the above example shows that this statement is false). The diagram shows that the differential equation $dy/dx = 1/x$ does have solutions which fill up the two regions $x > 0$ and $x < 0$ in which the equation makes sense; however the *formulae* needed in the two regions are quite distinct. It is therefore necessary to specify a function, not only by a formula, but also by adding a precise statement of the values of x for which the formula holds.



The aim of Chapter 1 is simply to introduce the minimum amount of language necessary to enable such precise statements to be made. In Chapter 2 a large number of examples are discussed from the commonsense viewpoint adopted above for the equation $dy/dx = 1/x$; in each example the method is first to draw slopes illustrating what the equation means, then to guess a formula for a solution in each region where the equation makes sense, and finally to check that there is a solution through each point of the region. The more formal discussion begins in Chapter 3.

The reader who, as a matter of principle, reads only the introduction of any book, may like to amuse himself by considering the following examples, which I owe to Ronald Brown. Here and throughout the book, $\sqrt{x}$ denotes the *positive* square root of a real number $x \geqslant 0$.

1 What is wrong with the following examination question? Find the least value of the function y such that $dy/dx = x - (1/x)$ and $y = 1$ when $x = 1$.

2 It is often said that the general solution of the equation $dy/dx = \sqrt{(1 - y^2)}$ is $y = \sin\ (a + x)$. This is false since for this it is not always true that $dy/dx \geqslant 0$. Make a correct statement.

3 Why not attempt to evaluate $\int_{-1}^{1} \left(\sin \frac{1}{x}\right)^2 dx$ by substituting $x = 1/u$?

4 Differentiate $\sin^{-1}(2 + x^2)$ and $\sqrt{(1 - x^2)} + \sqrt{(x^2 - 1)}$. Comment.

5 An A-level question asks for the general solution of $dy/dx = y/x$. Show why it is *not* $y = ax$ but

$$y = ax, \qquad x > 0$$

$$y = bx, \qquad x < 0$$

where a, b are real numbers.

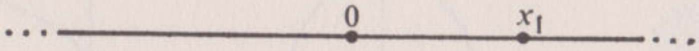
6 Find a solution of $dy/dx = 1 + 2(y/x)$ which has a minimum at $x = 1$ and a maximum at $x = -\frac{1}{2}$.

CHAPTER ONE

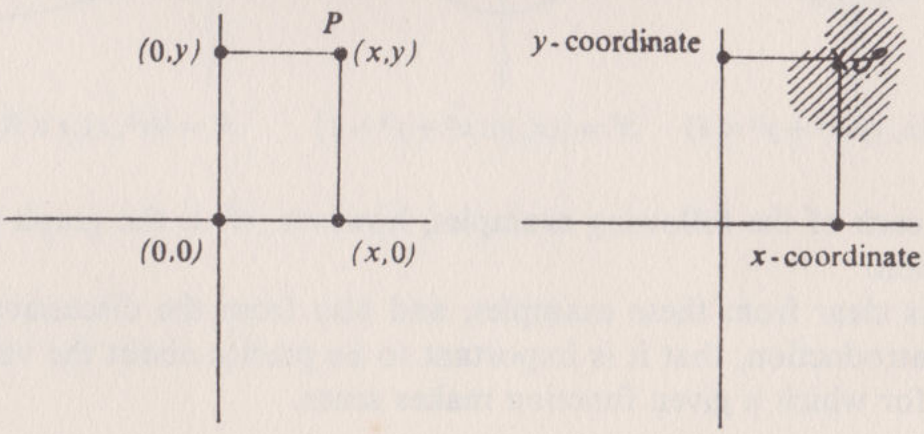
Differentiable functions

We take for granted the existence of the real numbers, with all their usual properties. Individual real numbers may be denoted by symbols like 0, 1, 2, 1·33 . . . , $\frac{1}{2}$, π or by symbols like t, x, y, x_1, x_2, ... We do not regard 'infinity' as a real number, nor do we allow symbols like t, x, y to denote complex values unless this is explicitly stated.

In practice it is often preferable to think not in terms of algebraic formulae but in terms of geometric pictures. In order to do so, consider the set **R** of *all* real numbers, and the set $\mathbf{R}^2$ of *all pairs* of real numbers. We think of **R** as an infinite line, and of a particular real number x_1 as a point on the line.



In the same way we think of $\mathbf{R}^2$ as a plane, and of a particular



pair (x_1, y_1) of real numbers as a point P on the plane. We call x_1 the *x-coordinate* and y_1 the *y-coördinate* of the point P.

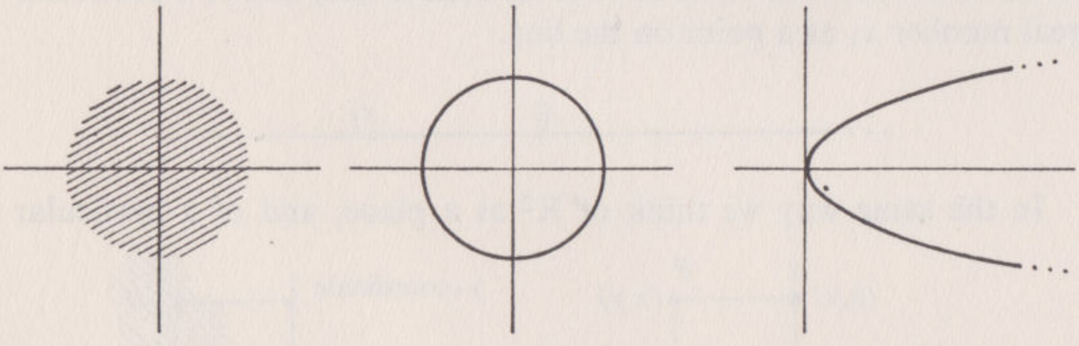
Some regions $\mathscr{S}$ in $\mathbf{R}^2$ act as graphs determining y_1 uniquely in terms of x_1. The condition for this is that $\mathscr{S}$ should meet each vertical line in not more than one point.

1.1 Definition

A region $\mathscr{S}$ in $\mathbf{R}^2$ is the *graph of a function* if no two points in $\mathscr{S}$ have the same x-coordinate.

1.2 Examples

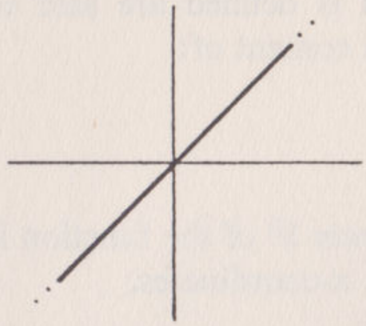
We use two slightly different notations to specify regions in $\mathbf{R}^2$. For example $\mathscr{S} = \{(x, y): x^2 + y^2 < 1\}$ means that $\mathscr{S}$ consists of all points (x, y) whose coordinates satisfy the inequality $x^2 + y^2 < 1$. Similarly $\mathscr{S} = \{(x, y): x^2 + y^2 = 1\}$ means that $\mathscr{S}$ consists of all points (x, y) whose coordinates satisfy $x^2 + y^2 = 1$. In this notation we write $\mathscr{S} = \{(x, y)\text{: some relation between } x \text{ and } y\}$. The other notation specifies both coordinates in terms of a parameter. Thus $\mathscr{S} = \{(t^2, t): t \in \mathbf{R}\}$ means that $\mathscr{S}$ consists of all points (t^2, t) where t is an arbitrary real number. In fact none of these three examples are graphs of functions, since in each case there are distinct points in $\mathscr{S}$ with the same x-coordinate.



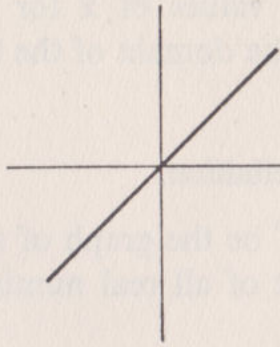
$\mathscr{S} = \{(x, y): x^2 + y^2 < 1\}$ $\quad$ $\mathscr{S} = \{(x, y): x^2 + y^2 = 1\}$ $\quad$ $\mathscr{S} = \{(t^2, t): t \in \mathbf{R}\}$

In each of the following examples, however, $\mathscr{S}$ is the graph of a function.

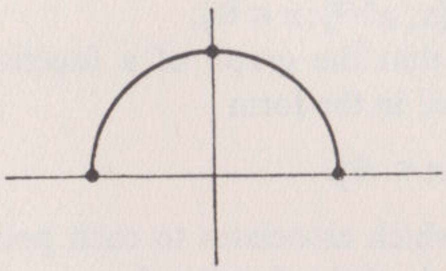
It is clear from these examples, and also from the discussion in the Introduction, that it is important to be precise about the values of x for which a given function makes sense.



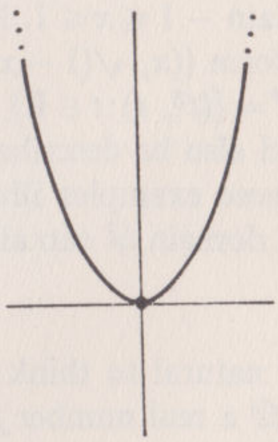
$\mathscr{S} = \{(t, t) : t \in \mathbf{R}\}$



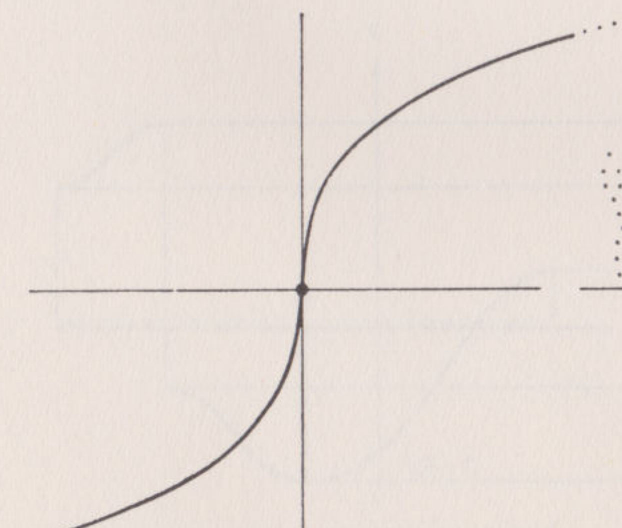
$\mathscr{S} = \{(t, t) : -1 \leqslant t \leqslant 1\}$



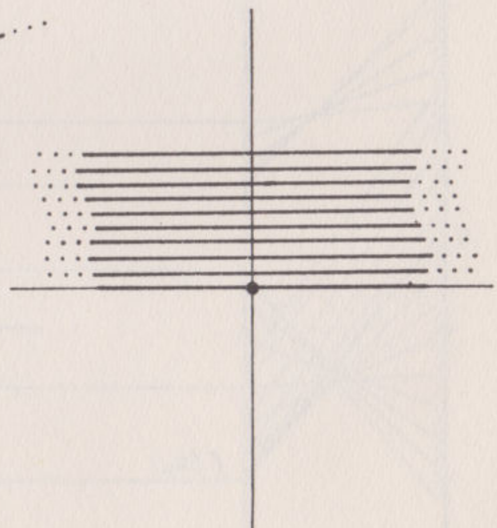
$\mathscr{S} = \{(x, y) : x^2 + y^2 = 1 \text{ and } y \geqslant 0\}$



$\mathscr{S} = \{(x, y) : y = x^2\}$



$\mathscr{S} = \{(t^3, t) : t \in \mathbf{R}\}$



$\mathscr{S} = \{(x, y) : y = 19^{\text{th}}$ digit in decimal expansion of $x\}$

The values of x for which the function is defined are said to form the domain of the function. This is the content of:

1.3 Definition

Let $\mathscr{S}$ be the graph of a function. The *domain* $\mathscr{D}$ of the function is the set of all real numbers which occur as x-coordinates.

1.4 Examples

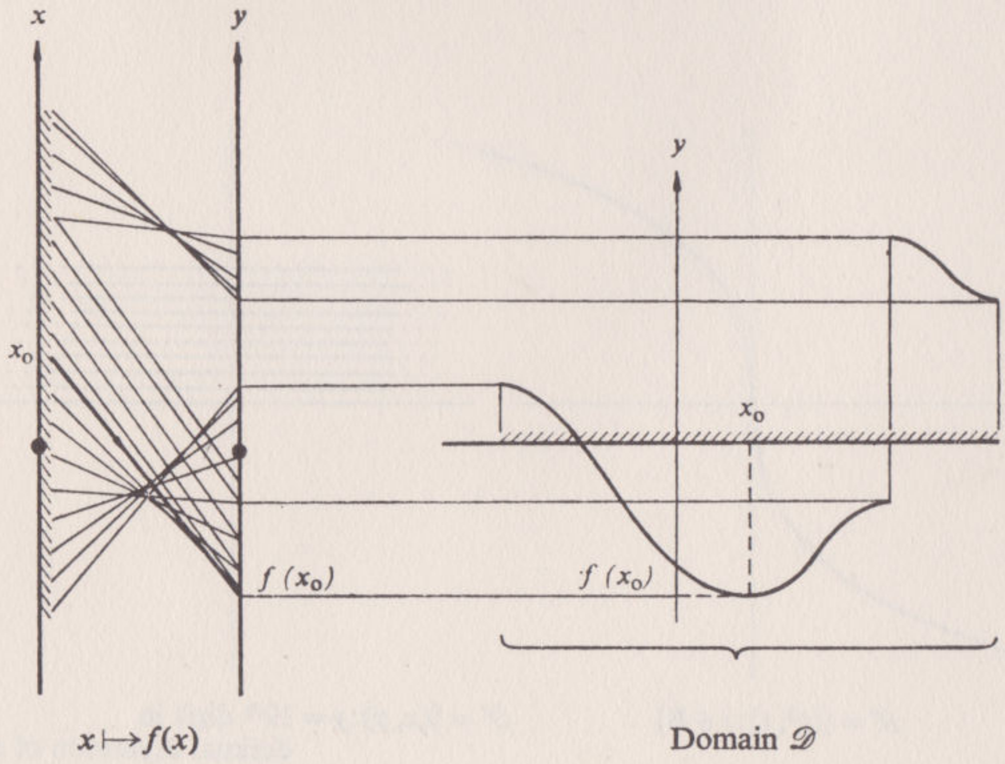
$\mathscr{S} = \{(x, y) : x^2 + y^2 = 1 \text{ and } y \geqslant 0\}$ is the graph of a function with domain $-1 \leqslant x \leqslant 1$. Note that this graph could also be described in the form $\{(x, \sqrt{(1 - x^2)}) : -1 \leqslant x \leqslant 1\}$.

$\mathscr{S} = \{(t^3, t) : t \in \mathbf{R}\}$ is the graph of a function with domain $\mathbf{R}$. It could also be described in the form $\{(x, x^{1/3}) : x \in \mathbf{R}\}$.

These examples illustrate the fact that the graph of a function with domain $\mathscr{D}$ can always be described in the form

$$\mathscr{S} = \{(x, f(x)) : x \in \mathscr{D}\}.$$

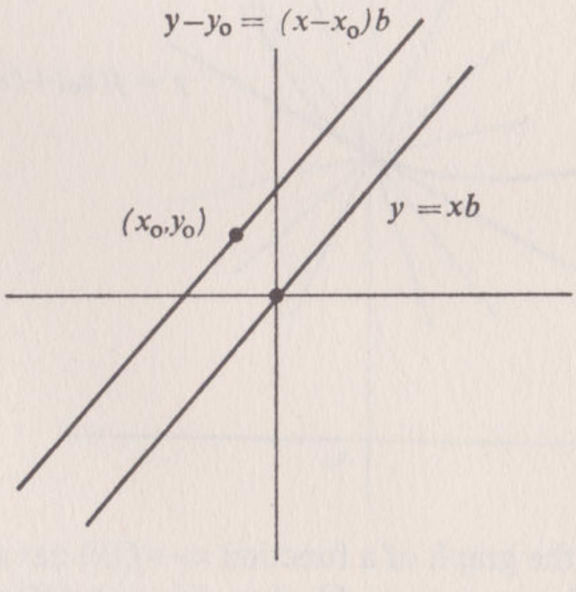
It is natural to think of f as a rule which associates to each point $x \in \mathscr{D}$ a real number $f(x)$, and to speak of the *function* f.



We now have two ways of describing a function. As a graph: this leads to the common notation $y = f(x)$, $x \in \mathscr{D}$. As a rule which associates a real number $f(x)$ to each point $x \in \mathscr{D}$: this leads to the less familiar notation $x \mapsto f(x)$, $x \in \mathscr{D}$. The use of the special arrow $\mapsto$ is to avoid confusion with other contexts (e.g. $\to$ meaning 'tends to'). These two descriptions are illustrated together: for some purposes the idea of a graph $y = f(x)$ is more convenient, for other purposes the idea of a rule $x \mapsto f(x)$ is preferable. It will often be the case that $f(x)$ is specified by an algebraic formula (for example $f(x) = x^2, f(x) = x$), or that f is given a particular name (for example $f(x) = \sin x, f(x) = \log x, f(x) = \exp x$).

1.5 Examples

A *linear function* has a graph of the form $\{(x, xb) : x \in \mathbf{R}\}$ for some real number b. We speak of the function $y = xb$ or $x \mapsto xb$. Note that under a linear function $0 \mapsto 0$. An *affine function* has a graph of the form $\{(x, a + xb) : x \in \mathbf{R}\}$ for some real numbers a, b. We speak of the function $y = a + xb$ or $x \mapsto a + xb$. If $y_0 = a + x_0 b$ the graph can also be written in the form $\{(x, y_0 + (x - x_0)b) : x \in \mathbf{R}\}$ or $y - y_0 = (x - x_0)b$. In both these examples the real number b is called the *slope* of the function.



We recall that certain functions can be differentiated or integrated.

1.6 Differentiation

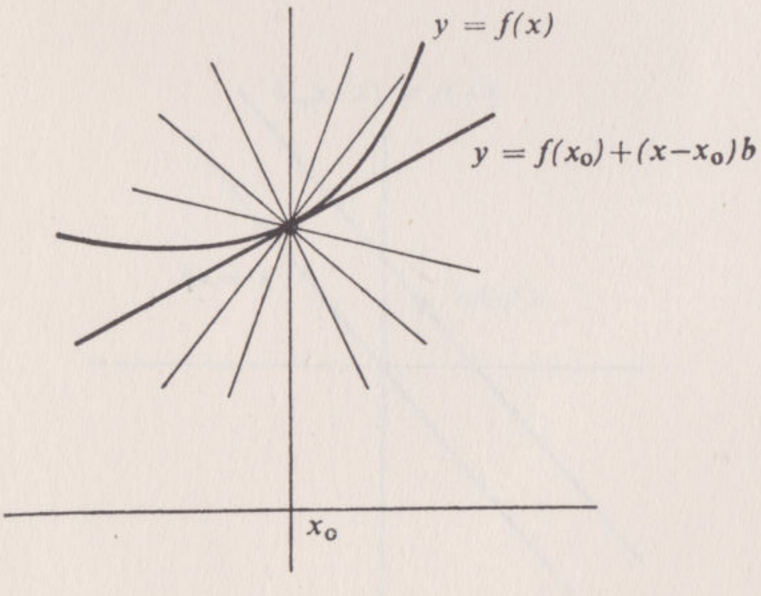
Certain functions $x \mapsto f(x)$, $x \in \mathscr{D}$ have a *derivative* $x \mapsto f'(x)$, $x \in \mathscr{D}$. The real number $b = f'(x_0)$ has the following significance. It is the *slope* of the affine function $x \mapsto f(x_0) + (x - x_0)b$ which gives the closest approximation to $x \mapsto f(x)$ near x_0. If this approximation exists for all $x_0 \in \mathscr{D}$ then the function $x \mapsto f(x)$ is said to be *differentiable*.

1.7 Remarks

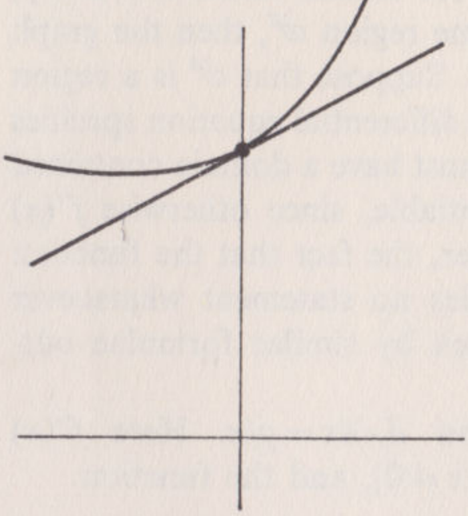
To make the above description of differentiation into a precise definition requires a precise explanation of the words 'closest approximation' and 'near' in 1.6. This requires the concept of a *continuous function* as given for instance in Moss–Roberts, Chapter 2 (especially 2.1 and 2.4). The geometric idea behind the definition of differentiation is that if $x \mapsto f(x)$ is differentiable then each point $(x_0, f(x_0))$ of the graph has a *non-vertical tangent*. Every non-vertical line through $(x_0, f(x_0))$ is the graph of an affine function

$$x \mapsto f(x_0) + (x - x_0)b$$

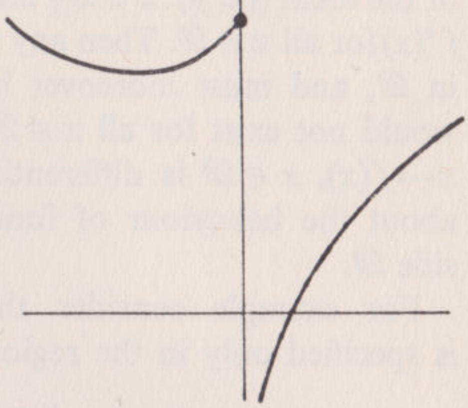
for some real number b. Among these the tangent is the line giving the closest approximation to $f(x)$.



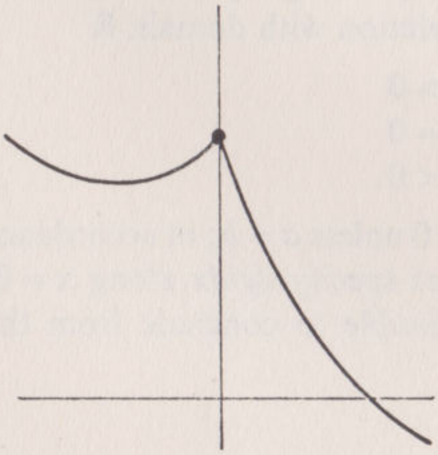
Note that, if the graph of a function $x \mapsto f(x)$ has a vertical tangent at $(x_0, f(x_0))$ for some $x_0 \in \mathscr{D}$, then f is *not* differentiable. In this case the tangent is not the graph of an affine function.



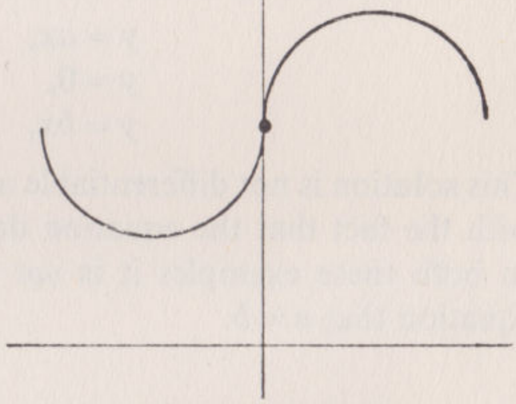
differentiable



not differentiable



not differentiable



not differentiable

1.8 Integration

A function $x \mapsto F(x)$, $x \in \mathscr{D}$ such that $F'(x) = f(x)$ for all $x \in \mathscr{D}$ is called an *integral* of the function f. If such an F exists we say that the function $x \mapsto f(x)$ or $y = f(x)$ is *integrable*.

To every method of denoting the function $x \mapsto f(x)$ there corresponds a notation for differentiation and integration. Thus we may write $y = f(x)$ and $dy/dx = f'(x)$. It is also common to write Df or df/dx for f'. A *differential equation* arises typically when a relationship is given between y and dy/dx, or equivalently, between $f(x)$ and $f'(x)$. A function $x \mapsto f(x)$ which satisfies this relationship is called a *solution* of the differential equation.

If the original equation is given only for certain values of x and y, for example for all points (x, y) in some region $\mathscr{S}$, then the graph of any solution will be contained in $\mathscr{S}$. Suppose that $\mathscr{S}$ is a region of the form $\{(x, y): x \in \mathscr{D}\}$ and that the differential equation specifies $f'(x)$ for all $x \in \mathscr{D}$. Then any solution must have a domain contained in $\mathscr{D}$, and must moreover be differentiable, since otherwise $f'(x)$ would not exist for all $x \in \mathscr{D}$. However, the fact that the function $x \longmapsto f(x)$, $x \in \mathscr{D}$ is differentiable implies no statement whatsoever about the behaviour of functions given by similar formulae outside $\mathscr{D}$.

For example consider the equation $dy/dx = y/x$. Here $f'(x)$ is specified only in the region $\{(x, y): x \neq 0\}$, and the function

$$\begin{aligned} y &= ax, \qquad x > 0 \\ y &= bx, \qquad x < 0 \end{aligned}$$

is a solution. The equation $x(dy/dx) = y$ is slightly different: now $x = 0$ implies $y = 0$, so that there is a solution with domain $\mathbf{R}$

$$\begin{aligned} y &= ax, \qquad x > 0 \\ y &= 0, \qquad x = 0 \\ y &= bx, \qquad x < 0. \end{aligned}$$

This solution is not differentiable at $x = 0$ unless $a = b$; in accordance with the fact that the equation does not specify dy/dx along $x = 0$. In both these examples it is *not* permissible to conclude from the equation that $a = b$.

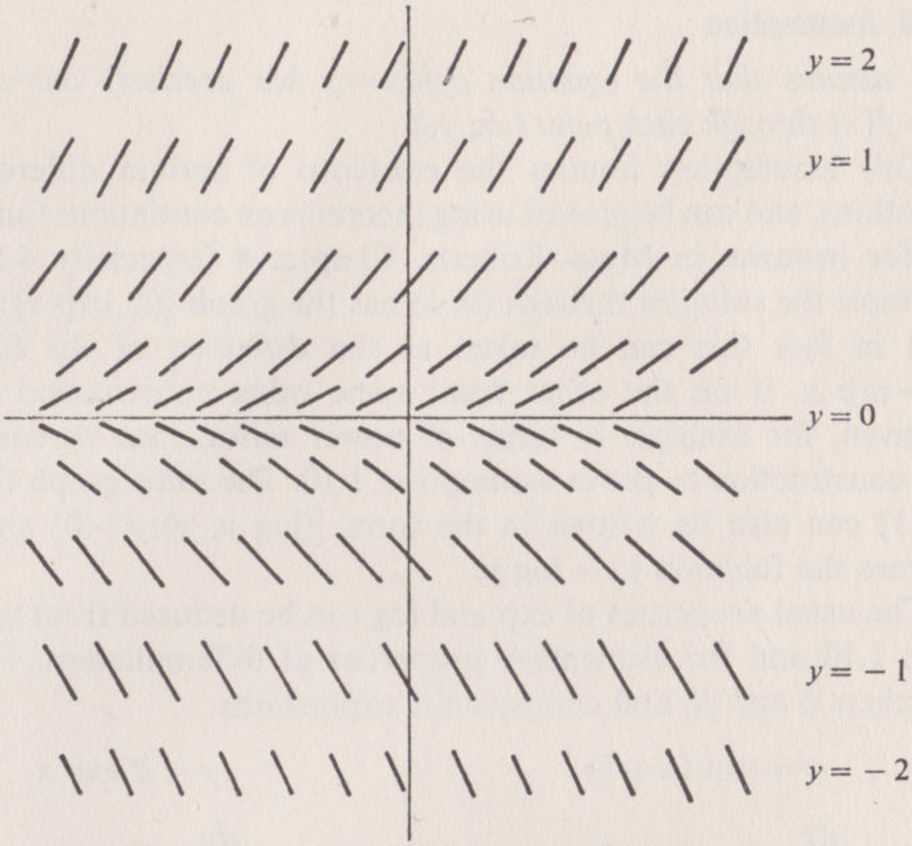
1.9 Example

Consider the equivalent equations

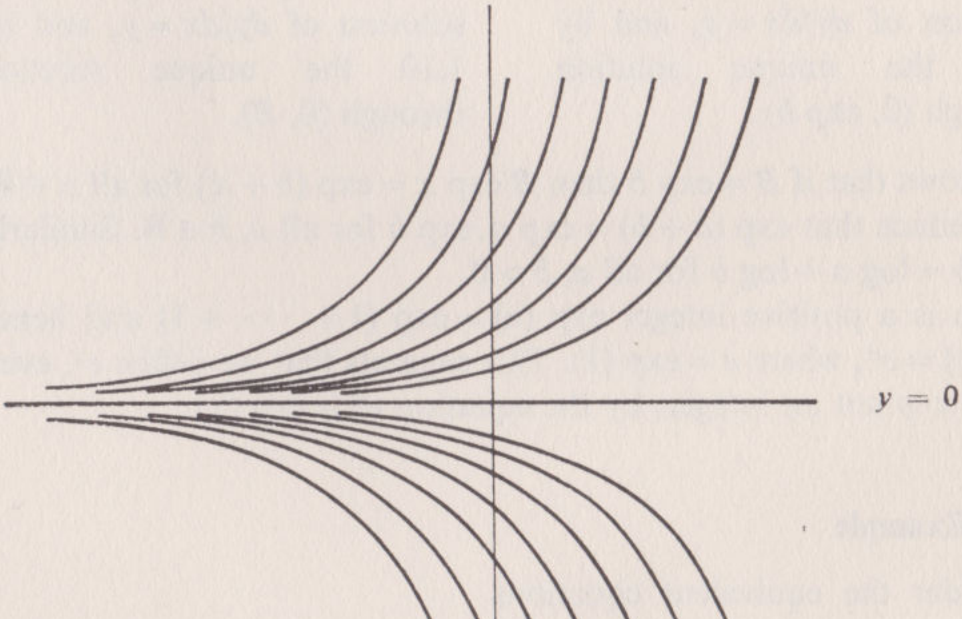
$$f'(x) = f(x) \quad \text{or} \quad \frac{dy}{dx} = y.$$

The equation is given without restrictions on x, y; that is the relation holds for all $(x, y) \in \mathbf{R}^2$. Any solution must be a differentiable function $x \longmapsto f(x)$. A solution with graph containing the point (x_0, y_0) must have slope y_0 there. The required slopes can be illustrated on a diagram.

Clearly $f(x) = 0$ is one solution. It is plausible from the diagram, but not obvious, that there is a pattern of curves with the required slopes, and that each such curve is a solution. It is also plausible that there is a unique such solution through each point (x_0, y_0).



y = 2
y = 1
y = 0
y = − 1
y = − 2



y = 0

1.10 Assumption

We assume that the equation $dy/dx = y$ has precisely one solution $x \mapsto f(x)$ through each point (x_0, y_0).

This assumption implies the existence of certain differentiable functions, and can be proved using theorems on continuous functions, as for instance in Moss–Roberts, Chapter 4 (especially 4.5). For example the solution through (0, 1) has the graph $\{(x, \exp x) : x \in \mathbf{R}\}$ and in fact this can be taken as the *definition* of the function $x \mapsto \exp x$. If on the other hand some other construction of exp is given, for example in terms of power series, then we could use the construction to prove assumption 1.10. The same graph through (0, 1) can also be written in the form $\{(\log y, y) : y > 0\}$ and thus defines the function $y \mapsto \log y$.

The usual properties of exp and log can be deduced from assumption 1.10 and the elementary properties of differentiation. Fix real numbers b and B, and compare the expressions

$y = \exp(x+b)$	$y = B \exp x$
$\dfrac{dy}{dx} = \exp(x+b)$	$\dfrac{dy}{dx} = B \exp x$
Therefore $y = \exp(x+b)$ is a solution of $dy/dx = y$, and by 1.10 the unique solution through $(0, \exp b)$.	Therefore $y = B \exp x$ is a solution of $dy/dx = y$, and by 1.10 the unique solution through $(0, B)$.

It follows that if $B = \exp b$ then $B \exp x = \exp(b + x)$ for all $x \in \mathbf{R}$. We deduce that $\exp(a+b) = \exp a . \exp b$ for all $a, b \in \mathbf{R}$. Similarly $\log ab = \log a + \log b$ for all $a, b > 0$.

If n is a positive integer $\exp(n) = \exp(1 + \cdots + 1)$ and hence $\exp(n) = e^n$, where $e = \exp(1)$. This suggests that we define e^x, even when x is not an integer, by the equation $e^x = \exp(x)$.

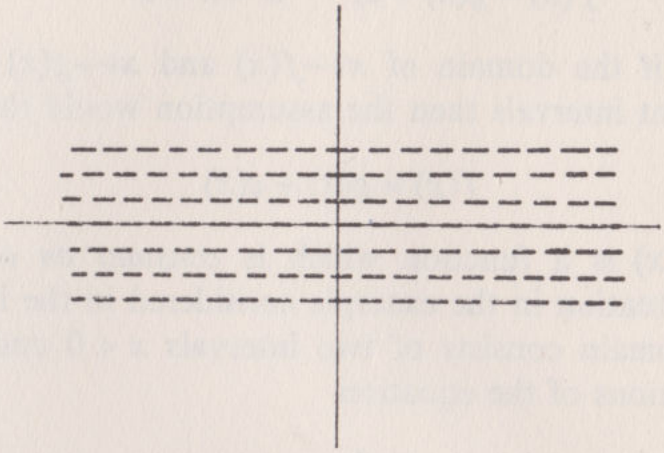
1.11 Example

Consider the equivalent equations

$$f'(x) = 0 \quad \text{or} \quad dy/dx = 0.$$

The equation is given without restrictions on x, y; that is the relation holds for all $(x, y) \in \mathbf{R}^2$. Any solution must be a differentiable

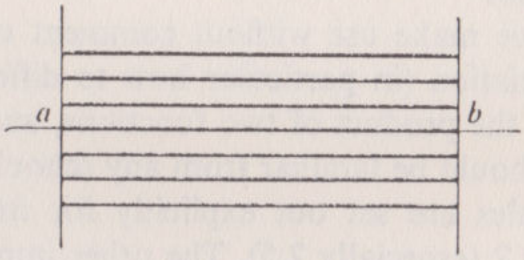
function $x \mapsto f(x)$ with domain **R**. The required slopes can be illustrated on a diagram.



Clearly for any real number c there is a solution $y = c$. Similarly, if I denotes the interval $a < x < b$ the equation

$$f'(x) = 0, \qquad x \in I$$

has solutions $f(x) = c$ with domain I. It is plausible from the diagram, but not obvious, that these are the only differentiable functions with domain I which satisfy the equation.



1.12 Assumption

We assume that if $x \mapsto f(x)$ is a differentiable function with domain $a < x < b$ and $f'(x) = 0$ for all $a < x < b$, then there is a real number c such that f is the constant function $f(x) = c$.

This assumption can be proved using theorems on differentiable functions, as for instance in Moss–Roberts, Chapter 3 (especially 3.4). It implies the uniqueness of certain differentiable functions. Suppose $x \mapsto f(x)$ and $x \mapsto g(x)$ are two differentiable functions with domain $a < x < b$ such that $f'(x) = g'(x)$. Then the function $x \mapsto h(x)$

defined by $h(x) = f(x) - g(x)$ has domain $a < x < b$ and $h'(x) = 0$ for all $a < x < b$. Therefore by 1.12 there is a real number c such that

$$f(x) = g(x) + c, \qquad a < x < b.$$

Note that if the domain of $x \mapsto f(x)$ and $x \mapsto g(x)$ consisted of several disjoint intervals then the assumption would imply

$$f(x) = g(x) + c(x)$$

where $x \mapsto c(x)$ is a function which is *constant on each interval.* This is the situation in the example considered in the introduction, where the domain consists of two intervals $x < 0$ and $x > 0$, and any two solutions of the equation

$$f'(x) = \frac{1}{x}, \qquad x \neq 0,$$

differ by a function

$$c(x) = a, \qquad x > 0,$$

$$c(x) = b, \qquad x < 0.$$

It is necessary to use such functions $x \mapsto c(x)$ in order to make correct the traditional rules about 'arbitrary constants' mentioned in the Introduction.

In the sequel we make use without comment of the elementary rules for differentiation (in particular how to differentiate the sum of two functions, the product of two functions, and a function of a function) which should be familiar from any school treatment of the subject. These rules are set out explicitly for instance in Moss–Roberts, Chapter 2 (especially 2.5). The other important properties used in the sequel are those summarized in assumptions 1.10 and 1.12. Any other property which is needed, will be mentioned as it arises.

Exercises

1 Make use of the usual rules for differentiation to find derivatives for the functions $\sqrt{x}$, $\log x$, $\sin x$, $\log \sqrt{x}$, $\log \log x$, $\log \sin x$, $\log \log \sin x$.

In each case make a careful statement of the domain of the function, and the domain of its derivative.

2 Find integrals for the following functions with domains as indicated:

(i) $\dfrac{1}{\sqrt{x}}$ $\qquad x > 0,$

(ii) $\dfrac{1}{x}$ $\qquad x \neq 0,$

(iii) $\cos x$ $\qquad$ all x,

(iv) $\dfrac{\cos x}{\sin x}$ $\qquad x \neq n\pi$ for integer n,

(v) $\dfrac{1}{x \log x}$ $\qquad 0 < x < 1 \quad \text{or} \quad x > 1,$

(vi) $\dfrac{\cos x}{\sin x \log \sin x}$ $\qquad 2n\pi < x < (2n+1)\pi$ for integer n.

3 Sketch the graphs of the functions

$$\begin{aligned} &x \mapsto x, && x \in \mathbf{R}, \\ &x \mapsto (\sqrt{x})^2, && x \geqslant 0, \\ &x \mapsto \sqrt{(x^2)}, && x \in \mathbf{R}, \\ &x \mapsto \frac{x^2}{x}, && x \neq 0, \end{aligned}$$

(the four diagrams obtained should all be different).

CHAPTER TWO

Examples of differential equations

Frequently the functions which occur in a differential equation are functions of time. For this reason we change notations slightly, and consider graphs $\{(t, f(t)) : t \in \mathscr{D}\}$ of functions $x = f(t)$ or $t \mapsto f(t)$ with domain $\mathscr{D}$. Suppose some relationship between t, $f(t)$ and $f'(t)$ is given; in what circumstances do we expect the existence of a function $t \mapsto f(t)$ satisfying the relationship, and how large is the domain of the function? It is instructive to consider more examples like 1.9. In the first few examples the equation specifies $f'(t)$ in a certain region of the (t, x)-plane. We adopt the following procedure: firstly draw the lines of slope on a diagram, secondly sketch the curves with the given lines as tangents (the so-called *solution curves*), and thirdly use one of the curves to sketch the graph of a typical solution of the equation.

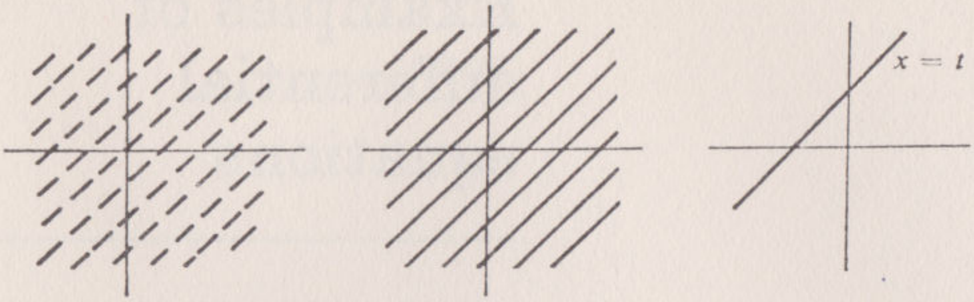
In many cases the shape of the curves suggests immediately how to guess the equation $x = f(t)$ of a solution. Once a guess has been made it is usually easy to check whether the function $x = f(t)$ is in fact a solution, and whether there is a solution through each point (t, x). The discussion of explicit methods for deducing the equation $x = f(t)$ of a solution is postponed until Chapter 3.

2.1 Example

$f'(t) = 1$, also written $dx/dt = 1$.

The equation asserts that the slope at each point is 1. The diagram suggests that every solution has the form $x = t + c$, where c is a

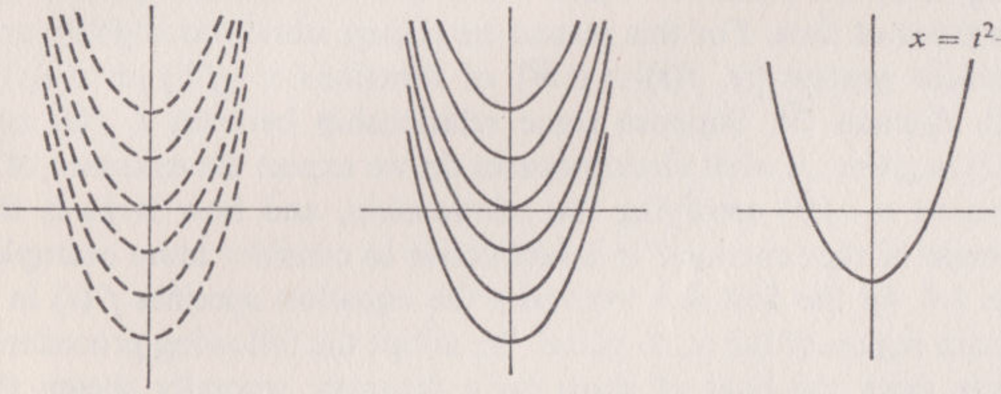
real number, and that there is a unique solution with domain **R** through each point. In fact both these statements follow from assumption 1.12.



Slope lines, solution curves and typical solution of $f'(t) = 1$

2.2 Example

$f'(t) = 2t$, also written $dx/dt = 2t$.



In this case there is again a unique solution with domain **R** through each point. Clearly $x = t^2$ satisfies the equation; therefore by assumption 1.12 every solution with domain **R** has the form $x = t^2 + c$ where c is a real number, and there is one such curve through each point (t, x).

2.3 Examples

Consider the differential equation

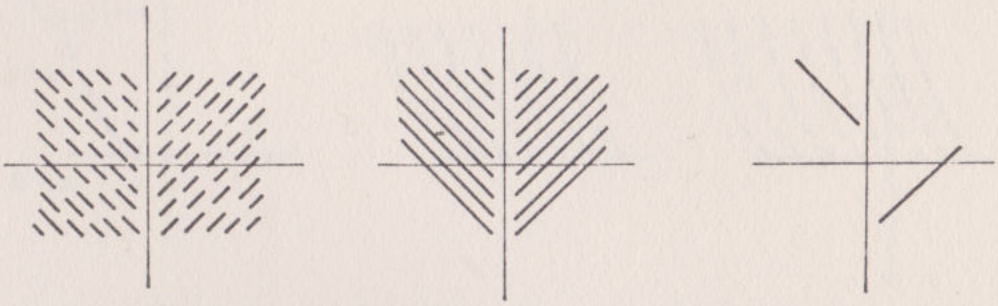
$$f'(t) = 1, \qquad t > 0$$
$$f'(t) = -1, \qquad t < 0.$$

In this case $f'(t)$ is undefined at $t = 0$, so that we expect solutions with domain $\mathbf{R}^* = \{t : t \neq 0\}$. In the region $t > 0$ there is a unique solution $x = t + a$, $t > 0$ through each point; in the region $t < 0$

there is a unique solution $x = -t + b$, $t < 0$ through each point. However, a typical solution with domain $\mathbf{R}^*$ has the form

$$\begin{aligned} x &= t + a, \qquad & t > 0 \\ x &= -t + b, \qquad & t < 0 \end{aligned}$$

where a, b are real numbers. Again these statements follow from assumption 1.12.



The situation is similar to the equation $f'(t) = 1/t$, $t \neq 0$ discussed in the introduction. Here a typical solution has the form

$$\begin{aligned} x &= \log t + a, \qquad & t > 0 \\ x &= \log(-t) + b, \qquad & t < 0 \end{aligned}$$

where a, b are real numbers.

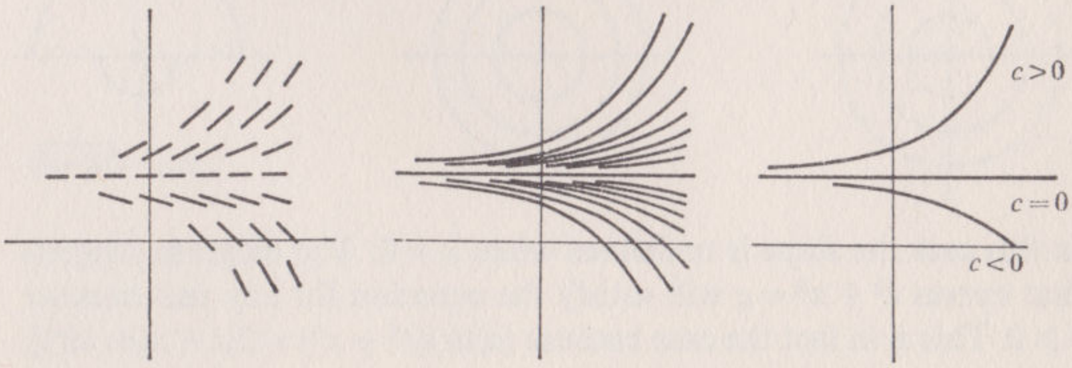
Notice that the equation $f'(t) = t/t$, $t \neq 0$ is *not* the same as example 2.1, since the slope is undefined at $t = 0$. A typical solution has the form

$$\begin{aligned} x &= t + a, \qquad & t > 0 \\ x &= t + b, \qquad & t < 0. \end{aligned}$$

This distinction can be important in applications, where an expression like t/t occurs in the middle of a calculation and is cancelled unwittingly.

2.4 Example

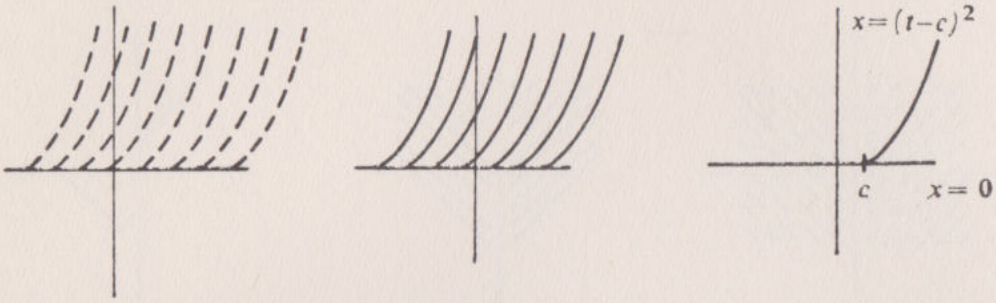
$f'(t) = f(t) - 2$, or $dx/dt = x - 2$.



The situation is almost the same as that in 1.9. By assumption 1.10 every solution has the form $x = 2 + ce^t$ where c is a real number, and there is a unique solution through each point (t, x).

2.5 Example

$f'(t) = 2\sqrt{f(t)}$, or $dx/dt = 2\sqrt{x}$.



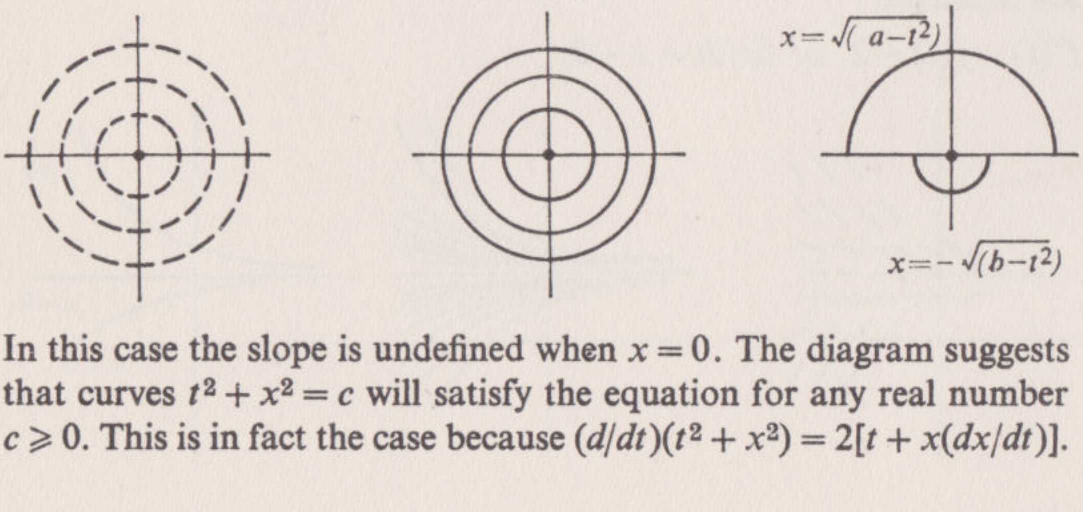
Here, and in the sequel, $\sqrt{}$ denotes positive square root. The equation implies that $x \geqslant 0$. In the region $x > 0$ there is a unique solution through each point. The function $x = (t-c)^2$, $t > c$ satisfies the equation, and has domain $t > c$ for some real number c. However, in the region $x \geqslant 0$ there are solutions of the form

$$\begin{aligned} x &= (t-c)^2, & t &\geqslant c \\ x &= 0, & t &\leqslant c \end{aligned}$$

with domain $\mathbf{R}$, as well as the solution $x = 0$. Note that through any point on the line $x = 0$ there pass an infinite number of solutions. The reader may suspect, correctly, that this is connected with the fact that at $x = 0$ the function $x \mapsto 2\sqrt{x}$ is not differentiable.

2.6 Example

$t + f(t)f'(t) = 0$, or $t + x(dx/dt) = 0$

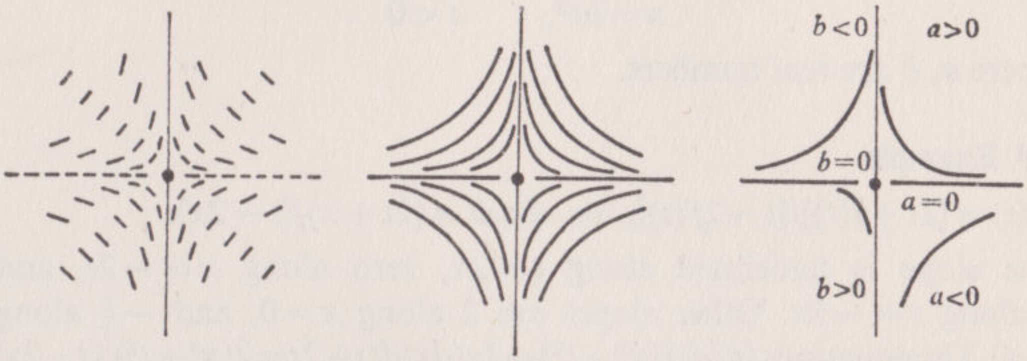


In this case the slope is undefined when $x = 0$. The diagram suggests that curves $t^2 + x^2 = c$ will satisfy the equation for any real number $c \geqslant 0$. This is in fact the case because $(d/dt)(t^2 + x^2) = 2[t + x(dx/dt)]$.

Thus typical solutions are $x=\sqrt{(a-t^2)}$ with domain $\{t:t^2<a\}$, $x=0$ with domain $\{t:t=0\}$, and $x=-\sqrt{(b-t^2)}$ with domain $\{t:t^2<b\}$, where a, b are positive real numbers. There is a unique solution through each point (t, x) with $x\neq 0$.

2.7 Example

$tf'(t)+f(t)=0$, or $x+t(dx/dt)=0$.



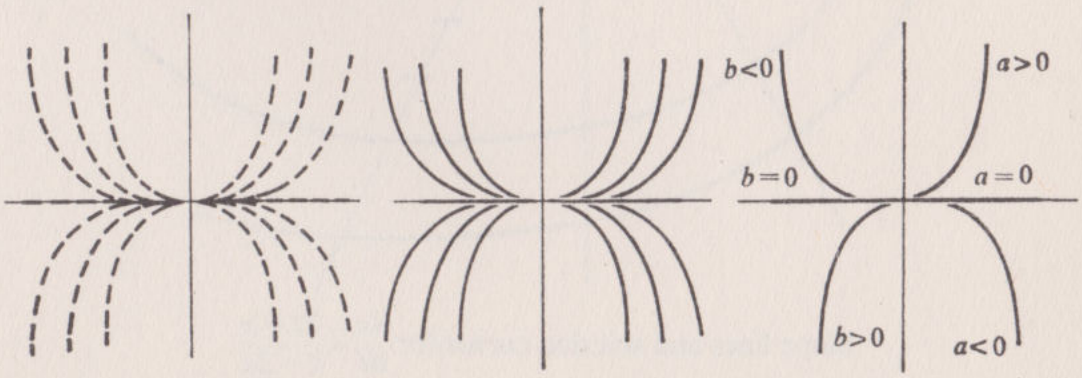
The slope is undefined when $t=0$. The diagram suggests that curves $tx=c$ will satisfy the equation for any real number c. This is in fact the case because $(d/dt)tx=x+t(dx/dt)$. Through each point of the region $t<0$ there is a unique solution of the form $x=b/t$ with domain $t<0$. Typical solutions with domain **R** have the form

$$\begin{aligned} x&=\frac{a}{t}, && t>0\\ x&=0, && t=0\\ x&=\frac{b}{t}, && t<0 \end{aligned}$$

where a, b are real numbers.

2.8 Example

$tf'(t)=3f(t)$, or $t(dx/dt)=3x$.



The slope is undefined when $t=0$. The diagram suggests that curves $x=ct^3$ will satisfy the equation for any real number c. This is in fact the case because $t(d/dt)(ct^3)=3ct^3$. Through each point of the region $t<0$ there is a unique solution of the form $x=bt^3$ with domain $t<0$. Typical solutions with domain **R** have the form

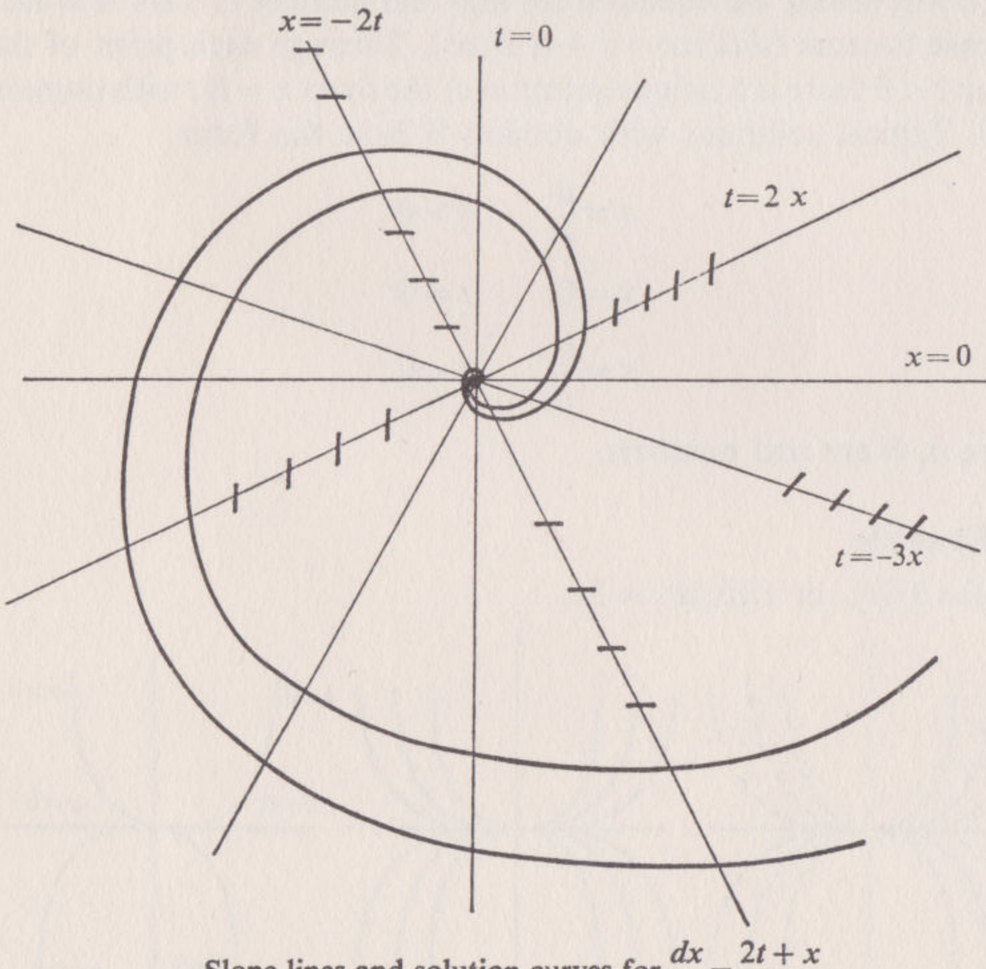
$$\begin{aligned} x&=at^3, & t&>0\\ x&=0, & t&=0\\ x&=bt^3, & t&<0 \end{aligned}$$

where a, b are real numbers.

2.9 Example

$f'(t)=[2t+f(t)]/[t-2f(t)]$, or $dx/dt=(2t+x)/(t-2x)$.

The slope is undefined along $t=2x$, zero along $x=-2t$, and 1 along $t=-3x$. Other slopes are 2 along $x=0$, and $-\frac{1}{2}$ along $t=0$. The equations $(d/dt)(x^2+t^2)=2x(dx/dt)+2t=2(x^2+t^2)/(t-2x)$ imply that as t increases the distance from (t,x) to $(0,0)$ increases



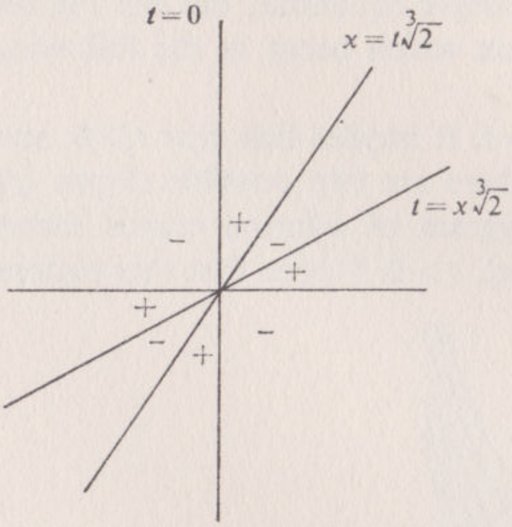
Slope lines and solution curves for $\dfrac{dx}{dt}=\dfrac{2t+x}{t-2x}$

or decreases according as $t > 2x$ or $t < 2x$. Therefore the curves spiral in to the origin in a clockwise direction as shown.

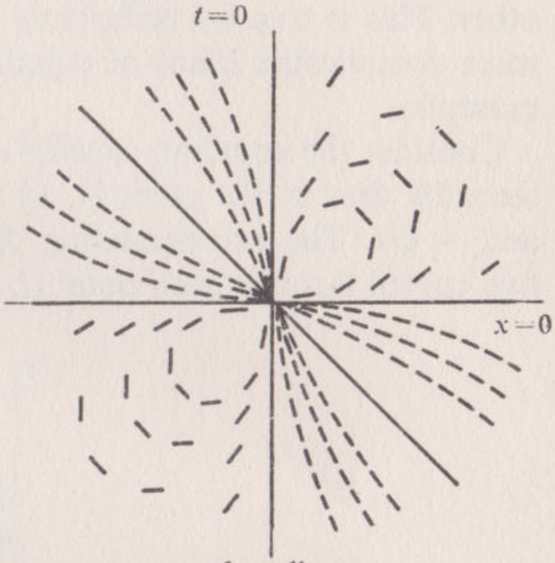
Typical solutions are given by fragments of spirals chosen so as to be graphs of functions (as in definition 1.1). In this case we make no attempt to guess a formula for x in terms of t. The reader may suspect, correctly, that to do so conveniently requires polar co-ordinates or complex numbers.

2.10 Example

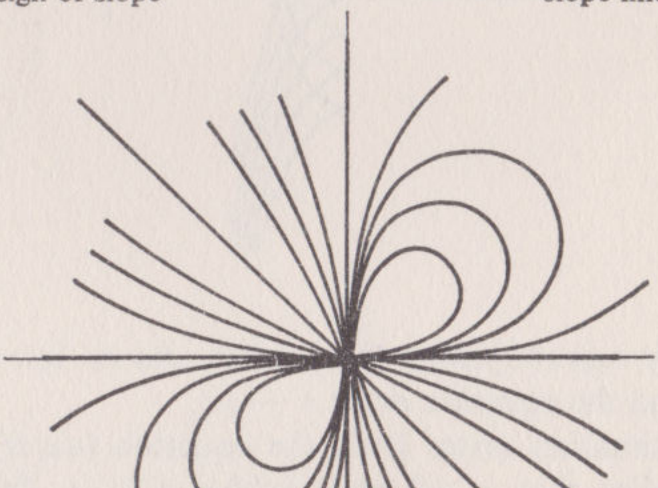
$t(2x^3 - t^3)(dx/dt) + x(2t^3 - x^3) = 0.$



sign of slope



slope lines



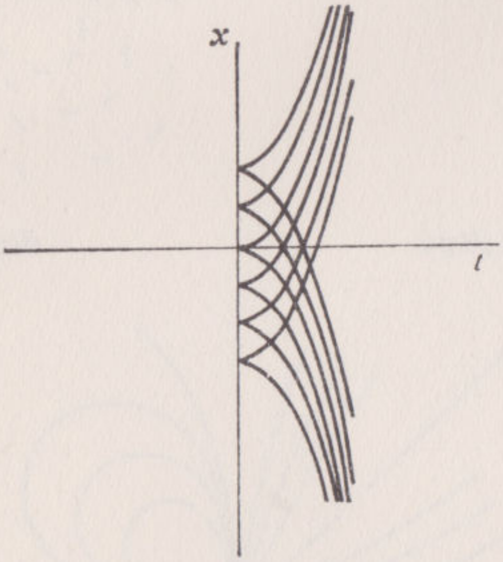
solution curves

The slope is undefined along $t=0$ and $t=x\sqrt{2}$, zero along $x=0$ and $x=t\sqrt{2}$, and -1 along $x=t$ and $x=-t$.

2.11 Examples

Examples 2.9 and 2.10 illustrate the fact that a sketch of the solution curves may well give useful and detailed information, even though it is not possible easily to write down formulae for the solution. It might be thought that every differential equation decomposes regions of the (t, x)-plane into curves which hardly ever cross each other. This is true for sufficiently simple equations, but not for the more complicated kinds of equation which occur in the following examples.

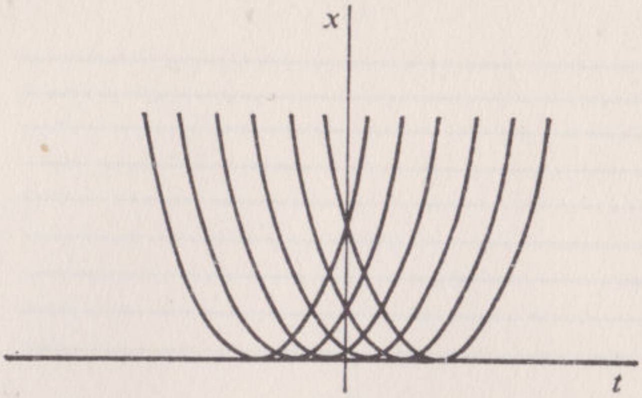
Consider the equation $(dx/dt)^2=t$. It implies first that $t \geqslant 0$, and secondly that at the point (t, x) there are two possible slopes $\sqrt{t}$ and $-\sqrt{t}$. The corresponding diagram of solution curves shows two curves through each point (t, x), $t>0$. Notice that this pattern



can be got by superimposing the solution curves for the equation $dx/dt=\sqrt{t}$ and the equation $dx/dt=-\sqrt{t}$.

A similar situation arises from the equation $(dx/dt)^2=4x$. The equation implies that $x \geqslant 0$ and $dx/dt= \pm 2\sqrt{x}$. In the region $x>0$ the solutions to $dx/dt=2\sqrt{x}$ (compare example 2.5) and $dx/dt=-2\sqrt{x}$ are superimposed.

Differential equations involving $(dx/dt)^2$ but no higher powers are said to be of the *second degree*. Such equations frequently determine a region of the (t, x)-plane in which there are no solutions,



and a region in which there are precisely two solutions through each point (obtained by superimposing solutions of two equations of first degree).

2.12 Examples

Consider the equation $(d^2x/dt^2) = 0$, or $f''(t) = 0$.

This equation involves the *second* derivative of $t \mapsto f(t)$, and is therefore said to be of *second order*. It is satisfied by any affine function

$$x = a + tb$$

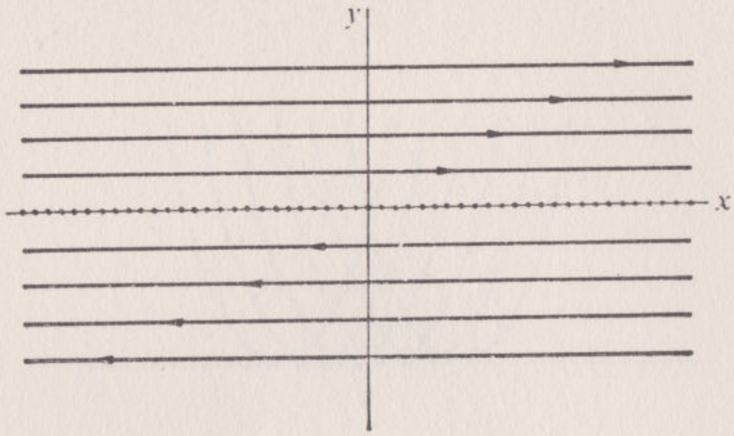
where a, b are real numbers. There is an infinite collection of solution curves through each point (t, x), since every non-vertical line in the (t, x)-plane is the graph of a solution, and hence there is no convenient diagram of solution curves.

A better approach to this equation is to substitute $y = dx/dt$, and to write the equation

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = 0.$$

Now a solution can be regarded as a pair of functions $t \mapsto x(t), y(t)$ and represented by a curve in the (x, y)-plane. Clearly the solution curves have the form $y = a$; the equation $dx/dt = y$ shows that x is increasing if $a > 0$, constant if $a = 0$, and decreasing if $a < 0$. Solutions are illustrated in the diagram overleaf: points on opposite sides of the line of rest points move in opposite directions. This method of replacing a second-order equation by two first-



order equations involving t, x, y, dx/dt, dy/dt is adopted in Chapter 4. In Chapter 3 we restrict attention to first-order equations.

Exercises

1 Sketch solution curves of the differential equations

$$\frac{dx}{dt} = 2xt$$

$$x\frac{dx}{dt} = t$$

$$t\frac{dx}{dt} = x.$$

2 Let $|t|$ denote the modulus of t, and $\surd$ positive square root. Explain the differences between the equations

$$\left(\frac{dx}{dt}\right)^2 = t,\ \left(\frac{dx}{dt}\right)^2 = |t|,\ \frac{dx}{dt} = \surd t,\ \frac{dx}{dt} = -\surd t,\ \frac{dx}{dt} = \surd|t|$$

and sketch solution curves for each (the five diagrams obtained should all be different).

3 Consider the differential equation

$$t\left(\frac{dx}{dt}\right)^2 - x\frac{dx}{dt} + 1 = 0.$$

Prove that there are no solutions through points in the region $\{(t, x);\ x^2 < 4t\}$. What can be said about the solution curves in the rest of the (t, x)-plane?

CHAPTER THREE

Solution of first-order equations

The method used to solve the differential equations of the previous chapter was essentially to guess the formula for a solution and then to check that it works. Often however it is possible to deduce the form of a solution by using elementary integration. This is particularly the case for equations of the *first order*, that is, involving $t, f(t)$ but no higher derivatives. We make use of the usual rules for integration (including substitution and integration by parts) as well as the standard properties of trigonometric and exponential functions. A careful discussion of all these matters is given in Moss–Roberts, Chapter 4 (especially 4.4, 4.5, 4.6).

3.1 Equations of the form $dx/dt = g(t)$

Examples 2.1–2.3 are of this form, in which dx/dt depends only on t and not on x. The examples illustrate that the domain of the function g may not be the whole line $\mathbf{R}$, and that to obtain a solution given by a single formula it is necessary to restrict to an interval $p < t < q$ in which g is well behaved.

We adopt the convention, when referring to an interval $p < t < q$, that the interval may in fact be unbounded at one end ($p < t$ or $t < q$) or at both ends (the whole line $\mathbf{R}$); however, we prefer not to use picturesque symbols like '$q = +\infty$' to describe this situation.

Theorem. *Let g be an integrable function with domain $p < t < q$. Then the differential equation*

$$\frac{dx}{dt} = g(t)$$

has a unique solution $x = f(t)$ through each point (t_0, x_0) with $p < t_0 < q$. The domain of f is the interval $p < t < q$.

Proof. Since g is integrable there is, by 1.8, a function G with domain $p < t < q$ such that $G'(t) = g(t)$. Let $c = x_0 - G(t_0)$. Then $(t) = c + G(t)$ gives the required solution. If $h(t)$ is another solution with $h(t_0) = x_0$, and $k(t) = f(t) - h(t)$, then k is a function with domain $p < t < q$ which satisfies the equations $k'(t) = 0$, $k(t_0) = 0$. By assumption 1.12 this implies $k(t) = 0$. Therefore the solution $x = f(t)$ is unique.

Remark. The lack of content in this proof is due to the fact that we defined integration in 1.8 as the opposite of differentiation. If integration were defined in some other way – for example by calculating areas or by integrating power series term by term – the proof would be more difficult and the theorem more significant. The application of this theorem depends on being able to compute the integral G in a specific case; we give some examples of this, using elementary facts about integration.

3.2 Examples

Let a, b be real numbers, and consider the equation

$$\frac{dx}{dt} = at + b.$$

By assumption 1.12 there is at most one function $t \mapsto G(t)$ with $G'(t) = at + b$ and $G(0) = 0$, so clearly it must be $t \mapsto \frac{1}{2}at^2 + bt$. Therefore every solution to the equation has the form

$$x = \tfrac{1}{2}at^2 + bt + c$$

where c is a real number.

Now consider the equation

$$\frac{dx}{dt} = \frac{1}{\sqrt{(1 - t^2)}}, \qquad -1 < t < 1.$$

By assumption 1.12 there is at most one function $t \mapsto G(t)$ with $G'(t) = 1/\sqrt{(1-t^2)}$ and $G(0) = 0$. We use the standard notation

$$G(t) = \int_0^t \frac{1}{\sqrt{(1-u^2)}}\, du$$

for this function. Since $-1 < t < 1$ we can write $u = \sin\theta$ with $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, and obtain

$$\int_0^t \frac{1}{\sqrt{(1-u^2)}}\, du = \int_0^{\sin^{-1} t} \frac{1}{\cos\theta} \cos\theta\, d\theta = \sin^{-1} t.$$

Here $\sin^{-1} t$ denotes the unique number θ such that $\sin\theta = t$ and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Thus every solution to the equation has the form

$$x = c + \sin^{-1} t, \qquad -1 < t < 1$$

where c is a real number.

Consider next the rather similar equation

$$\frac{dx}{dt} = \frac{1}{\sqrt{(t^2-1)}}, \qquad t < -1 \text{ or } t > 1.$$

Here the domain of the function $g(t) = 1/\sqrt{(t^2-1)}$ consists of two distinct intervals which must be considered separately. Define

$$\cosh v = \tfrac{1}{2}(e^v + e^{-v})$$
$$\sinh v = \tfrac{1}{2}(e^v - e^{-v}).$$

The properties of exp in 1.9 and 1.10 imply that $(d/dv)(\cosh v) = \sinh v$ and $(d/dv)(\sinh v) = \cosh v$. Moreover $(\cosh v)^2 - 1 = (\sinh v)^2$. If $u \geqslant 1$ there is a unique real number $v \geqslant 0$ such that $\cosh v = u$, and we write $v = \cosh^{-1} u$. These facts suggest the following substitutions.

(i) In the interval $u > 1$ let $u = \cosh v$, $v > 0$; then

$$\int_1^t \frac{1}{\sqrt{(u^2-1)}}\, du = \int_0^{\cosh^{-1} t} dv = \cosh^{-1} t$$

and so every solution has the form

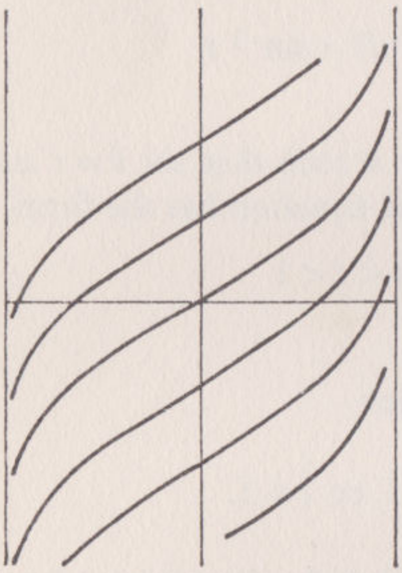
$$x = a + \cosh^{-1} t, \qquad t > 1.$$

(ii) In the interval $u < -1$ let $-u = \cosh v$, $v > 0$; then

$$\int_{-1}^{t} \frac{1}{\sqrt{(u^2-1)}}\, du = \int_{0}^{\cosh^{-1}(-t)} dv = \cosh^{-1}(-t)$$

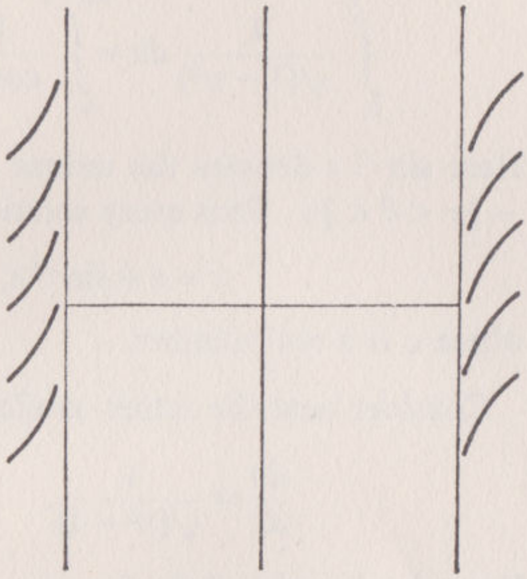
and so every solution has the form

$$x = b - \cosh^{-1}(-t), \qquad t < -1.$$



Solutions to

$$\frac{dx}{dt} = \frac{1}{\sqrt{(1-t^2)}}$$



Solutions to

$$\frac{dx}{dt} = \frac{1}{\sqrt{(t^2-1)}}$$

Consider the equation $dx/dt = 1/t$, $t \neq 0$. Again the domain of $g(t) = 1/t$ consists of two distinct intervals $t > 0$ and $t < 0$. By 1.10 every solution in the region $t > 0$ has the form $x = a + \log t$. Similarly every solution in the region $t < 0$ has the form $x = b + \log(-t)$.

The substitution $t - q$ for t, shows immediately that every solution of the equation $dx/dt = 1/(t-q)$ has the form

$$x = a + \log(t-q), \qquad q < t,$$

$$x = b + \log(q-t), \qquad t < q.$$

Finally consider the equation

$$\frac{dx}{dt} = \frac{1}{(p-t)(q-t)}, \qquad p < q.$$

The domain of $g(t) = 1/(p-t)(q-t)$ consists of three intervals $t < p$, $p < t < q$, and $q < t$. Since

$$\frac{1}{(p-t)(q-t)} = \frac{1}{q-p}\left(\frac{1}{t-q} - \frac{1}{t-p}\right)$$

every solution has the form

$$\begin{aligned} x &= c + \frac{1}{q-p}[\log(q-t) - \log(p-t)] \\ &= c + \frac{1}{q-p}\log\left(\frac{q-t}{p-t}\right) \quad \text{for } t < p, \\ x &= b + \frac{1}{q-p}[\log(q-t) - \log(t-p)] \\ &= b + \frac{1}{q-p}\log\left(\frac{q-t}{t-p}\right) \quad \text{for } p < t < q, \\ x &= a + \frac{1}{q-p}[\log(t-q) - \log(t-p)] \\ &= a + \frac{1}{q-p}\log\left(\frac{t-q}{t-p}\right) \quad \text{for } q < t. \end{aligned}$$

3.3 Equations of the form $dx/dt = h(x)$

Examples 2.4–2.5 are of this form, in which dx/dt depends only on x and not on t. The examples illustrate that the domain of h may not be the whole line $\mathbf{R}$, and that to obtain solutions given by a single formula it is necessary to restrict to an interval $a < x < b$ in which h is well behaved. Moreover it may be necessary to restrict to an even smaller interval to make sure that there is a unique solution through each point (t_0, x_0).

We first observe that if $h(c) = 0$ then $x = c$ is a solution. This makes it reasonable to restrict attention to intervals $a < x < b$ on which $h(x) > 0$ or $h(x) < 0$.

Theorem. *Let h be a function of constant sign with domain $a < x < b$ (that is, either $h(x) > 0$ for all $a < x < b$, or $h(x) < 0$ for all $a < x < b$). If $1/h$ is integrable, the differential equation*

$$\frac{dx}{dt} = h(x)$$

has a unique solution $x = f(t)$ through each point (t_0, x_0) with $a < x_0 < b$. The domain of f is some interval containing t_0.

Proof. Note first that by 3.1 the equation

$$\frac{dt}{dx}=\frac{1}{h(x)}, \qquad a<x<b$$

has a unique solution $t=k(x)$ through (t_0, x_0). Thus k is a function with domain $a<x<b$, and $k'(x)$ is of constant sign. We now use the inverse function theorem proved in Moss–Roberts, Chapter 3 (especially 3.5) to assert the existence of a unique function f, with domain an interval containing t_0, such that, for all $a<x<b$,

(i) $x=f(t)$ if and only if $t=k(x)$, and

(ii) $x=f(t-c)$ satisfies $dx/dt=h(x)$ if and only if $t=c+k(x)$ satisfies $dt/dx=1/h(x)$.

We conclude that every solution in the region $a<x<b$ has the form

$$x=f(t-c).$$

Note that in practice the formula for the inverse function f is usually immediate, so that the full strength of the inverse function theorem is not needed.

3.4 Examples

Consider the equation

$$\frac{dx}{dt}=\frac{1}{x}, \qquad x\neq 0.$$

A comparison with the proof of 3.3 shows that

$$\frac{1}{h(x)}=x$$

and therefore solutions of $dt/dx=x$ in the regions $x>0$ or $x<0$ have the form

$$t=c+\tfrac{1}{2}x^2.$$

Therefore through each point (t_0, x_0) with $x_0>0$ there is a solution of the form

$$x=\sqrt{(2t-2c)}$$

with domain $\{t\in\mathbf{R}: t>c\}$.

Consider next the equation $dx/dt = \frac{1}{2}(x^2 - 1)$. There are solutions $x = 1$ and $x = -1$ because for these values $h(x) = \frac{1}{2}(x^2 - 1)$ is zero. A comparison with the proof of 3.3 shows that

$$\frac{1}{h(x)} = \frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

and that therefore the solutions of $dt/dx = 1/h(x)$ in the regions $x < -1$ or $x > 1$ have the form

$$t = c + \log \frac{x-1}{x+1}.$$

This equation is equivalent to $e^{t-c} = (x-1)/(x+1)$, so that $1 + e^{t-c} = x(1 - e^{t-c})$. Through each point (t_0, x_0) with $x_0 > 1$ there is a solution of the form

$$x = \frac{1 + e^{t-c}}{1 - e^{t-c}} = \frac{1 + ae^t}{1 - ae^t}$$

where c is a real number and $e^{-c} = a > 0$. Note that the domain of this function is $\{t \in \mathbf{R} : t < -\log a\}$.

A similar calculation in the region $-1 < x < 1$ shows that $t = c + \log (1-x)/(1+x)$, and hence

$$x = \frac{1 - e^{t-c}}{1 + e^{t-c}} = \frac{1 - ae^t}{1 + ae^t}$$

is a solution with domain $\mathbf{R}$ for any $a > 0$.

The solutions to this equation are illustrated in 4.1.

Now consider the equation $dx/dt = \sqrt{(1 - x^2)}$, $-1 \leqslant x \leqslant 1$. There are solutions $x = 1$ and $x = -1$ because for these values $h(x) = \sqrt{(1 - x^2)}$ is zero. A comparison with the proof of 3.3 shows that

$$\frac{1}{h(x)} = \frac{1}{\sqrt{(1 - x^2)}}$$

and that therefore the solutions of $dt/dx = 1/h(x)$ in the region $-1 < x < 1$ have the form

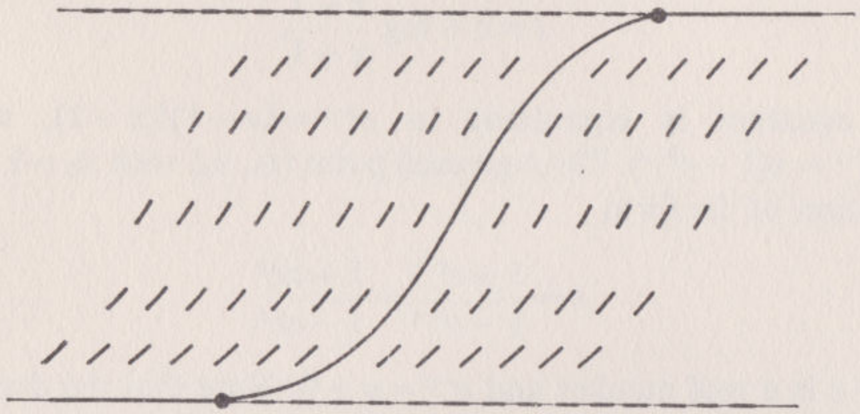
$$t = c + \sin^{-1} x, \qquad -1 < x < 1.$$

This equation is equivalent to

$$x = \sin (t - c), \qquad c - \tfrac{1}{2}\pi < t < c + \tfrac{1}{2}\pi,$$

and yields solutions with domain $c-\frac{1}{2}\pi<t<c+\frac{1}{2}\pi$. Through each point (t_0, x_0) with $-1<x_0<1$ there is a unique solution. However, through points $(t_0, -1)$ and $(t_0, 1)$ there are many solutions, for example every real number c determines a solution

$$\begin{array}{ll} x=-1 & t\leqslant c-\frac{1}{2}\pi, \\ x=\sin(t-c) & c-\frac{1}{2}\pi\leqslant t\leqslant c+\frac{1}{2}\pi, \\ x=1 & c+\frac{1}{2}\pi\leqslant t. \end{array}$$



For a further example consider the equation $dx/dt=3x^{2/3}$. Here $x^{2/3}$ denotes the unique cube root of the real number x^2. There is a solution $x=0$ because for this value $x^{2/3}$ is zero. Each point (t_0, x_0) in the region $x>0$ lies on a unique solution of the form $x=(t-q)^3$ with domain $t>q$. Each point in the region $x<0$ lies on a unique solution of the form $x=(t-p)^3$ with domain $t<p$. However if $p\leqslant q$ there are solutions with domain $\mathbf{R}$ of the form

$$x=\begin{cases}(t-p)^3 & t\leqslant p\\ 0 & p\leqslant t\leqslant q\\ (t-q)^3 & q\leqslant t\end{cases}$$

and so uniqueness fails for points on $x=0$.

3.5 Equations reducible to the form 3.1 or 3.3

The first-order equations considered in 3.1 and 3.3 are extremely special. A more typical equation would have the form

$$\frac{dx}{dt}=f(t, x), \qquad (t, x)\in\mathscr{S}$$

where $\mathscr{S}$ is a region in the plane $\mathbf{R}^2$ and $(t, x) \mapsto f(t, x)$ denotes a function which associates, to each point (t, x) of $\mathscr{S}$, a point $f(t, x)$ of $\mathbf{R}$. Occasionally, however, it is possible by cunning manipulation to reduce the above equation to one of the form 3.1 (in which $f(t, x)$ does not depend on x) or of the form 3.3 (in which $f(t, x)$ does not depend on t). We give some examples.

Consider an equation of the form

$$\frac{dx}{dt} = g(t)h(x)$$

where g is a function with domain $p < t < q$, and h is a function with domain $a < x < b$. If there are solutions for the equations

$$\frac{dy}{dt} = g(t), \qquad p < t < q, \text{ as in 3.1}$$

$$\frac{dx}{dy} = h(x), \qquad a < x < b, \text{ as in 3.3}$$

then $x = x(y(t))$ is a solution of the original equation inside the rectangle $p < t < q$, $a < x < b$. For instance the equation

$$\frac{dx}{dt} = 2tx$$

can be solved in this way. The equations

$$\frac{dy}{dt} = 2t$$

$$\frac{dx}{dy} = x$$

have solutions $y = t^2 + c$ and $x = be^y$ respectively. Therefore a solution to the original equation $dx/dt = 2tx$ is

$$x = be^{t^2+c} = (be^c)e^{t^2} = ae^{t^2}$$

where $a \in \mathbf{R}$. Equations of this type are sometimes said to have *variables separable.*

Consider now an equation of the form

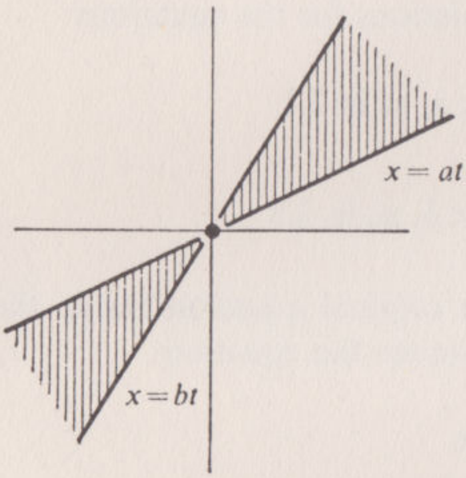
$$\frac{dx}{dt} = g(u)$$

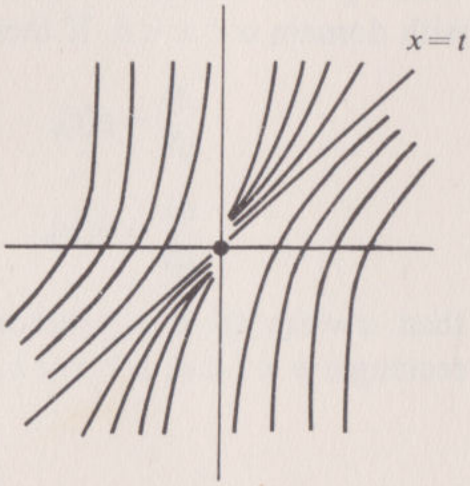
where $u = x/t$ and g is a function with domain $a < u < b$. This equation is therefore defined for all points (t, x) in the region strictly between the lines $x = at$ and $x = bt$. Note that (0, 0) is excluded, so that the region consists of two distinct parts. If $x = ut$ we have

$$g(u) = \frac{dx}{dt} = \frac{du}{dt}\, t + u$$

and hence an equation

$$\frac{du}{dt} = \frac{1}{t}\,(g(u) - u)$$





Solutions to

$$2\frac{dx}{dt} = \frac{x^2 + t^2}{t^2}$$

which is of the 'variables separable' type (note that $t \neq 0$, so that $1/t$ is well defined, throughout the region in question). For example the equation

$$2\,\frac{dx}{dt} = \frac{x^2 + t^2}{t^2}$$

is of this type, since $x = ut$ gives

$$2t\,\frac{du}{dt} + 2u = u^2 + 1$$

$$\frac{du}{dt} = \frac{1}{2t}\,(u-1)^2.$$

The equation $dy/dt = 1/2t$ has solutions of the form

$$y = \tfrac{1}{2}\log|\,t\,| + c$$

in the regions $t>0$ and $t<0$. The equation $du/dy=(u-1)^2$ has a solution $u=1$, as well as solutions $u=1-1/(y+q)$. Every point (x_0, t_0) with $t_0 \neq 0$ lies on a solution of the original equation given by a formula of the following type (we set $a=2(c+q)$).

If $x_0 > t_0 > 0$, $x = t - \dfrac{2t}{a+\log t}$ with domain $0 < t < e^{-a}$.

If $x_0 = t_0 > 0$, $x = t$ with domain $0 < t$.

If $t_0 > x_0 > 0$, $x = t - \dfrac{2t}{a+\log t}$ with domain $e^{-a} < t$.

If $x_0 < t_0 < 0$, $x = t - \dfrac{2t}{a+\log(-t)}$ with domain $-e^{-a} < t < 0$.

If $x_0 = t_0 < 0$, $x = t$ with domain $t < 0$.

If $t_0 < x_0 < 0$, $x = t - \dfrac{2t}{a+\log(-t)}$ with domain $t < -e^{-a}$.

The reader should use these formulae to make an accurate version of the rough sketch of solutions on the opposite page. Equations of this type are said to be *homogeneous*, but this is only one of the many distinct meanings of the word 'homogeneous' in connection with differential equations.

3.6 Exact equations

As in 3.5 we consider a function $(t, x) \mapsto f(t, x)$ which associates, to each point $(t, x) \in \mathscr{S}$, a point $f(t, x) \in \mathbf{R}$. We use without proof the fact that certain such functions are differentiable, and yield 'partial derivatives' $\dfrac{\partial f}{\partial t}$ and $\dfrac{\partial f}{\partial x}$. Thus $t \mapsto \dfrac{\partial f}{\partial t}(t, x_0)$ is defined to be the derivative of $t \mapsto f(t, x_0)$, for fixed x_0. Similarly $x \mapsto \dfrac{\partial f}{\partial x}(t_0, x)$ is the derivative of $x \mapsto f(t_0, x)$, for fixed t_0.

If $x=g(t)$ then $f(t, x)$ can be regarded as a function of t with derivative

$$\frac{d}{dt}(f(t, x)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt}.$$

This fact can often be exploited as follows. Consider a differential equation of the form

$$a(t, x) + b(t, x)\frac{dx}{dt} = h(t).$$

If it is possible to find a function $(t, x) \mapsto f(t, x)$ such that $\frac{\partial f}{\partial t}(t, x) = a(t, x)$ and $\frac{\partial f}{\partial x}(t, x) = b(t, x)$ then the equation can be written

$$\frac{d}{dt}(f(t, x)) = h(t)$$

and solved by the method of 3.1. We give two examples. Consider the equation

$$(pt + qx) + (qt + sx)\frac{dx}{dt} = 0$$

where p, q, s are real numbers. It can be written in the form

$$\frac{d}{dt}(\tfrac{1}{2}pt^2 + qtx + \tfrac{1}{2}sx^2) = 0$$

and therefore there are solution curves of the form

$$pt^2 + 2qtx + sx^2 = c$$

for any real number c. Fragments of these curves are graphs of solutions $x = g(t)$ of the original equation.

Note that an attempt to apply the same method to the equation

$$(pt + qx) + (rt + sx)\frac{dx}{dt} = 0$$

will fail if $r \neq q$. This illustrates the fact that it is only in certain lucky situations that the above method can be used.

For another example consider the equation

$$2x\frac{dx}{dt} + x^2 \cos t = 2t\, e^{-\sin t}.$$

It can be written in the form

$$x^2 e^{\sin t} \cos t + 2xe^{\sin t}\frac{dx}{dt} = 2t$$

$$\frac{d}{dt}(x^2 e^{\sin t}) = 2t$$

and therefore there are solution curves of the form

$$x^2 = (t^2 + c)\, e^{-\sin t}$$

for any real number c. The fragments of these curves given by

$$x = e^{-\frac{1}{2}\sin t} \sqrt{(t^2 + a)}, \qquad t^2 \geqslant -a,$$
$$x = -e^{-\frac{1}{2}\sin t} \sqrt{(t^2 + b)}, \quad t^2 \geqslant -b,$$

are solutions to the original equation. The method used in this example consisted of multiplying through by a non-zero 'integrating factor' $e^{\sin t}$, designed to make the equation exact. There is one class of differential equations for which this method always works; these are the so-called *linear equations.*

3.7 Linear equations

A first-order equation is *linear* if it can be written in the form

$$\frac{dx}{dt} + x\, g(t) = h(t)$$

where g, h are functions with domain $p < t < q$. We wish to investigate under what circumstances there exists a unique solution through each point (t_0, x_0) with $p < t_0 < q$.

We first assume g integrable. Then there is a unique function G with domain $p < t < q$ such that $G'(t) = g(t)$ and $G(t_0) = 0$. Since $e^{G(t)}$ is non-zero for all $p < t < q$, the differential equation is not changed by multiplying through by the integrating factor $e^{G(t)}$. Therefore the original equation is equivalent to

$$x\, e^{G(t)}\, g(t) + e^{G(t)} \frac{dx}{dt} = e^{G(t)}\, h(t)$$

$$\frac{d}{dt}(x\, e^{G(t)}) = h(t)\, e^{G(t)}.$$

We now assume he^G integrable. Then there is a unique function k with domain $p < t < q$ such that $k'(t) = h(t)\, e^{G(t)}$ and $k(t_0) = 0$. By 3.1 there is a unique solution through (t_0, x_0) of the form

$$x\ e^{G(t)} = x_0 + k(t)$$
$$x = e^{-G(t)}\,(x_0 + k(t)).$$

The solution is a function with domain $p < t < q$.

Remarks. The usual notation for G and k in terms of integrals is

$$G(t) = \int_{t_0}^{t} g(u)\,du,$$

$$k(t) = \int_{t_0}^{t} e^{G(u)}\,h(u)\,du.$$

The most important special case is when g is constant. Then the equation has the form

$$\frac{dx}{dt} + xa = h(t)$$

for some real number a. Every solution can be written in the form

$$x = e^{-at}\left(c + \int_{t_0}^{t} e^{au}\,h(u)\;du\right)$$

where c is a real number; this solution passes through (t_0, x_0) if $c = x_0 e^{at_0}$.

The theory of 3.1, 3.3, 3.5 and 3.7 may raise the following question. Clearly it is of interest to know whether a given differential equation has a solution through a given point (t_0, x_0). But why is it of interest to know whether or not the solution is unique? There are two main reasons. Firstly, in applications where x is a variable depending on time t, it is important to know whether or not the differential equation and the value x_0 at time t_0 determine x uniquely for all $t > t_0$. Secondly, differential equations are often solved in practice by programming a computer to make successive approximations to the solution; if the solution is not unique the computer may have a nervous breakdown trying to find it.

3.8 Equations reducible to linear form

Occasionally an equation which is not linear can be reduced to a linear equation by a suitable trick. For example consider the 'Bernoulli equation'

$$\frac{dx}{dt} + x\,g(t) = x^n\,h(t), \qquad x \neq 0.$$

Let $y = x^{1-n}$, which makes sense because $x \neq 0$. Then

$$\frac{dy}{dt} = (1-n)x^{-n}\frac{dx}{dt} = (1-n)h(t) - (1-n)x^{1-n}\, g(t)$$

so that y is a solution of the differential equation

$$\frac{dy}{dt} - (n-1)y\, g(t) = -(n-1)h(t)$$

which is linear.

For another example consider the equation

$$t\frac{dx}{dt} + x\, g(t) = h(t), \qquad t > 0.$$

Let $t = e^u$, which makes sense because $t > 0$. Then

$$\frac{dx}{du} = \frac{dx}{dt}\frac{dt}{du} = \frac{dx}{dt}\, t$$

so that x is a solution of the differential equation

$$\frac{dx}{du} + x\, g(e^u) = h(e^u)$$

which is linear.

In both these examples the substitution which makes possible the reduction to a linear equation depends crucially on the restrictions imposed ($x \neq 0$ in the first example; $t > 0$ in the second).

3.9 Examples

Consider the equation

$$\frac{dc}{dx} = \frac{c}{x}, \qquad x > 0.$$

This arises in economics, where c is the total cost of an output x, dc/dx is the so-called marginal cost, and c/x is the so-called average cost. This equation is homogeneous. The substitution $c = ux$ gives $du/dx = 0$ and hence every solution has the form $u = \text{constant}$. It follows that if the marginal cost equals average cost at every output then the average cost is constant.

Consider the equation $L(dx/dt) + xR = v(t)$. This arises from an

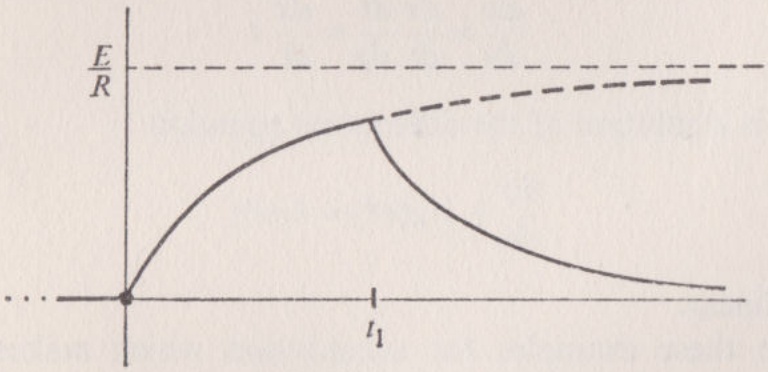
electric circuit with inductance $L>0$, resistance $R>0$, and voltage $v(t)$ at time t. The solution $x(t)$ is then the current at time t. This equation is linear. The method of 3.7 gives, writing $a=R/L$,

$$\frac{dx}{dt}+xa=\frac{1}{L}v(t),$$

$$\frac{d}{dt}(xe^{at})=\frac{1}{L}v(t)e^{at},$$

$$xe^{at}=c+\frac{1}{L}\int_0^t v(u)e^{au}\,du.$$

Suppose for example that $v(t)$ is a constant input E started at time $t=0$ and switched off at time $t=t_1$. There are then two cases.



For $0\leqslant t\leqslant t_1$ choose the solution with $x=0$ at $t=0$. This corresponds to choosing $c=0$ and $v(u)=E$. Therefore

$$x\,e^{at}=\frac{E}{L}\int_0^t e^{au}\,du=\frac{E}{La}(e^{at}-1)$$

and hence

$$x=\frac{E}{R}(1-e^{-Rt/L})\qquad\text{for } 0\leqslant t\leqslant t_1.$$

For $t_1\leqslant t$ choose the solution with $x=(E/R)(1-e^{-at_1})$ at $t=t_1$. This corresponds to choosing $c=(E/R)(e^{at_1}-1)$ and $v(u)=0$. Therefore

$$x=\frac{E}{R}(e^{Rt_1/L}-1)\,e^{-Rt/L}\qquad\text{for } t_1\leqslant t.$$

Exercises

1 Find every solution $t \mapsto x(t)$ of the following differential equations, as well as the solution through (1, 0).

(i) $\dfrac{dx}{dt} = 3t^2 + 4t,$ (ii) $\dfrac{dx}{dt} = be^t, \quad b \in \mathbf{R},$

(iii) $\dfrac{dx}{dt} = \dfrac{1}{1+t^2},$ (iv) $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{(1+t^2)}},$

(v) $\dfrac{dx}{dt} = \cos t,$ (vi) $\dfrac{dx}{dt} = \dfrac{\cos t}{\sin t}, \quad t \neq n\pi.$

2 For each of the following differential equations find every solution $t \mapsto x(t)$, and state its domain. Is the solution through (0, 1) unique?

(i) $\dfrac{dx}{dt} = x^2 - 3x + 2,$ (ii) $\dfrac{dx}{dt} = be^x, \quad b \in \mathbf{R},$

(iii) $\dfrac{dx}{dt} = (x-1)^2,$ (iv) $\dfrac{dx}{dt} = \sqrt{(x^2-1)}, \quad x \geqslant 1,$

(v) $\dfrac{dx}{dt} = 2\sqrt{x}, \quad x \geqslant 0,$ (vi) $\dfrac{dx}{dt} = \tan x, \quad x \neq (n+\frac{1}{2})\pi.$

3 Find all solutions of the differential equations

(i) $(3t^2x - tx) + (2t^3x^2 + t^3x^4)\dfrac{dx}{dt} = 0,$

(ii) $(1+2x) + (4-t^2)\dfrac{dx}{dt} = 0, \quad t > 2,$

(iii) $\dfrac{dx}{dt} = \cos\dfrac{x}{t}, \quad t > 0,$

(iv) $(t^2 - x^2)\dfrac{dx}{dt} = tx, \quad t > x > 0,$

(v) $e^{3t}\dfrac{dx}{dt} + 3xe^{3t} = 2t,$

(vi) $(2t+3x) + (3t-x)\dfrac{dx}{dt} = t^2.$

4 Solve the differential equations

(i) $\dfrac{dx}{dt} + 2x = e^t$,

(ii) $\dfrac{dx}{dt} + x \tan t = 0, \qquad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$,

(iii) $\dfrac{dx}{dt} - x \tan t = 4 \sin t, \qquad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$,

(iv) $t^3 \dfrac{dx}{dt} + (2 - 3t^2)x = t^3, \qquad t > 0$,

(v) $\dfrac{dx}{dt} + 2tx + tx^4 = 0, \qquad x > 0$,

(vi) $t \dfrac{dx}{dt} + x \log t = t^2, \qquad t > 0$.

5 Show how to solve an equation of the form

$$t \frac{dx}{dt} + x\, g(t) = h(t), \qquad t < 0$$

by a substitution $t = -e^u$.

6 Show that $x = t^3$ and $x = t^4$ are solutions of the differential equation

$$t^2 \frac{d^2x}{dt^2} - 6t \frac{dx}{dt} + 12x = 0.$$

Deduce that if p, q, r, s are real numbers, and

$$f(t) = \begin{cases} pt^3 + qt^4 & t \geqslant 0 \\ rt^3 + st^4 & t \leqslant 0 \end{cases}$$

then $x = f(t)$ is a solution. What restrictions are imposed on p, q, r, s if it is required that f be

(i) twice differentiable,

(ii) three times differentiable,

(iii) four times differentiable?

CHAPTER FOUR

Autonomous systems

In the previous two chapters solutions to a differential equation

$$\frac{dx}{dt}=f(t, x)$$

were considered as functions $t \mapsto x(t)$, or as curves in the (t, x)-plane. This description has certain disadvantages:

(i) unless the function f has an especially simple form it may not be possible to find explicit formulae for a solution;

(ii) such exact information may in any case not be required for the problem under discussion, and it might suffice to give a simpler description;

(iii) a physical problem might involve two functions x, y related by differential equations

$$\frac{dx}{dt}=g(t, x, y)$$

$$\frac{dy}{dt}=h(t, x, y)$$

so that solutions must now be represented by curves in the three dimensional space with coordinates (t, x, y).

The equations in (iii) may be imagined as controlling the motion of a particle in the (x, y)-plane. The velocity of the particle, if it reaches (x, y) at time t, has components dx/dt, dy/dt. In some physical

problems the velocity depends only on (x, y), not on the time t; that is, the direction and speed of travel depend only on the position (x, y) of the particle, not on the time at which it reaches that position. This situation corresponds to a pair of differential equations

$$\frac{dx}{dt} = g(x, y)$$

$$\frac{dy}{dt} = h(x, y)$$

where g, h are functions defined in a region $\mathscr{S}$ of $\mathbf{R}^2$. Such a pair is called an *autonomous system* in two variables. A solution of the autonomous system is a pair of functions $t \mapsto (x(t), y(t))$. In the same way, an equation

$$\frac{dx}{dt} = h(x)$$

of the type considered in 3.3, is called an *autonomous equation* (or an autonomous system in one variable). The essential feature, in both cases, is that the right-hand side does not depend on t.

The frequent occurrence of autonomous systems in two variables leads to a search for a simpler method of representing geometrically their solutions. First, we consider an example of the corresponding situation for one variable.

4.1 Examples

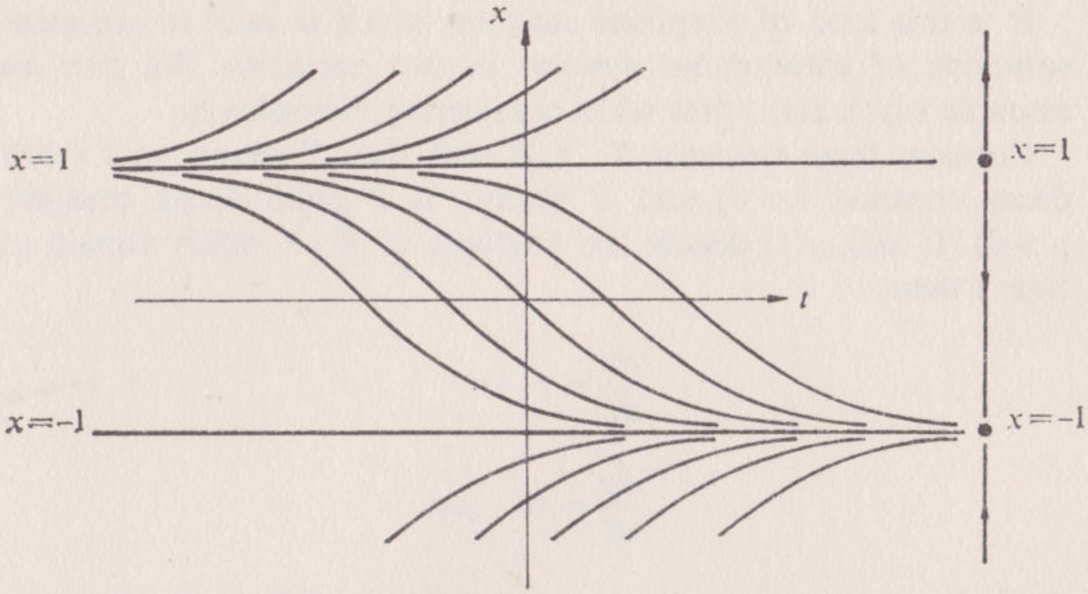
Consider the equation

$$\frac{dx}{dt} = \tfrac{1}{2}(x^2 - 1).$$

It is autonomous, because the right-hand side is independent of t, and explicit solutions were obtained in 3.4. For many purposes, however, the following analysis would be sufficient.

At $x = \pm 1$ we have $dx/dt = 0$, so $x = 1$ and $x = -1$ are rest points; for $x > 1$ or $x < -1$ we have $dx/dt > 0$, so at these points x increases with time; for $-1 < x < 1$ we have $dx/dt < 0$, so at these points x decreases with time. The left-hand side of the diagram represents the information given by the explicit formula: $x = 1$, $x = -1$ or

$$x = \frac{1 + Ae^t}{1 - Ae^t}, \qquad A \neq 0,$$



obtained in 3.4. The right-hand side of the diagram represents the information obtained in the cruder analysis above. Both diagrams display equally the most important facts about the equation, namely that $x=1$ is an *unstable* solution (points near $x=1$ move away from 1 as t increases), while $x=-1$ is a *stable* solution (points near $x=-1$ move towards -1 as t increases).

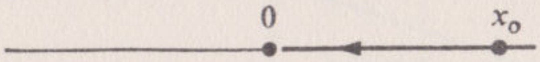
For a further example involving one variable, consider the equation

$$\frac{dx}{dt} = -\lambda x, \qquad x \geqslant 0.$$

This occurs in physics, where x denotes the amount of an element X subject to radioactive decay. The equation states that, at any given time t, the rate of decay dx/dt is proportional to the amount $x(t)$ which remains. Here λ is a positive real number, called the *decay constant*. If the amount of X at time $t=0$ is x_0, the equation has the unique solution

$$x = x_0 e^{-\lambda t}.$$

This solution can be represented by a single diagram



showing that x decreases to zero as t increases. Notice that since by definition $x \geqslant 0$, only the right half of the line is of interest.

It is this kind of simplified diagram which is used to represent solutions of autonomous systems in two variables. We give an example which also arises when considering radioactivity.

Consider three elements X, Y, Z such that X decays to Y (with decay constant $\lambda > 0$) and Y decays to Z (with decay constant $\mu > 0$). If $x(t)$, $y(t)$ denote the amounts of X, Y which remain at time t then

$$\frac{dx}{dt} = -\lambda x$$

$$\frac{dy}{dt} = \lambda x - \mu y.$$

As before, we represent x, y by a point in $\mathbf{R}^2$; notice however that since by definition $x \geqslant 0$ and $y \geqslant 0$, only the top right quadrant is of interest.

Now assume $\lambda < \mu$. What can be said about the relative values of x, y?

The equations imply that if $x \neq 0$

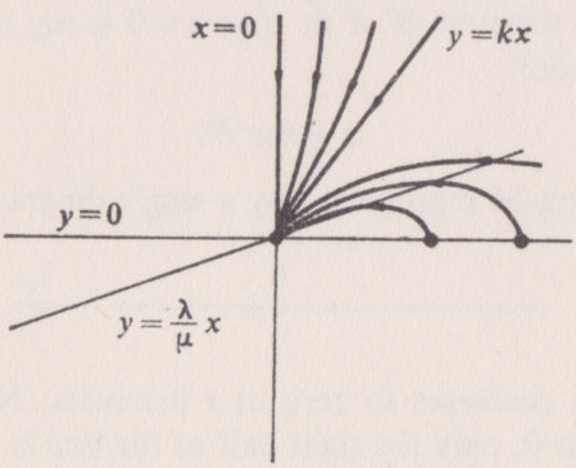
$$\frac{dy}{dx} = \frac{\lambda x - \mu y}{-\lambda x} = -1 + \frac{\mu}{\lambda}\frac{y}{x}.$$

Therefore the movement of the point (x, y) can be represented by curves with the following properties:

(i) along the line $y = 0$ each curve has slope -1;

(ii) between the lines $y = 0$ and $y = (\lambda/\mu)x$ the slope is negative;

(iii) between the lines $y = (\lambda/\mu)x$ and $x = 0$ the slope is positive.



Moreover there is a solution along the line $y = kx$ if and only if $k = -1 + (\mu/\lambda)k$; that is for

$$k = \frac{\lambda}{\mu - \lambda}.$$

There is also a solution $y = y_0 e^{-\mu t}$ along the line $x = 0$. These facts are summarized in the diagram. Notice in particular that if the initial position x_0, y_0 has $y_0 < (\lambda/\mu)x_0$ then y will at first increase and then decrease.

4.2 Phase space

A large class of examples of autonomous systems in two variables arise from second-order differential equations. Let $t \mapsto x(t)$ be a function representing the movement of a point along a line, and subject to a differential equation of the form

$$\frac{d^2x}{dt^2} = h\left(x, \frac{dx}{dt}\right)$$

where the function h is independent of t. Let $y = dx/dt$. Then $x(t)$ represents the position at time t, and $y(t)$ the velocity at time t. The above equation is equivalent to the autonomous system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = h(x, y).$$

It is now possible to plot solution curves in the (x, y)-plane, and to show that, if the function h satisfies suitable conditions, there is a unique solution through each point x_0, y_0. The (x, y)-plane, in which the distance x is plotted against the velocity y is often called *phase space*. The equation $dx/dt = y$ implies that as t increases, the point $(x(t), y(t))$ will move 'clockwise' along solution curves: from left to right in the top half $y > 0$ of the plane, and from right to left in the bottom half $y < 0$.

4.3 Examples

Consider the equation

$$\frac{d^2x}{dt^2} + px = 0, \qquad p > 0$$

which arises from harmonic motion. It can be written as an autonomous system

$$\frac{dx}{dt} = y$$

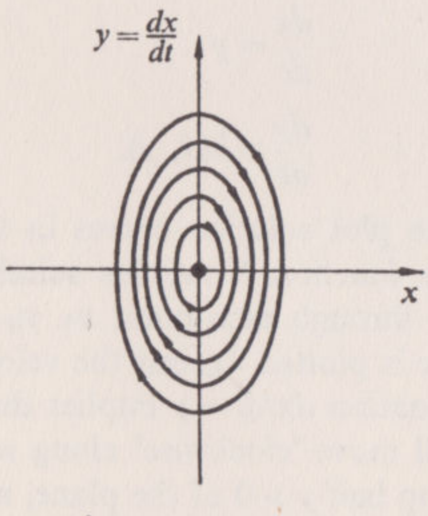
$$\frac{dy}{dt} = -px.$$

The equations imply that the origin $x = y = 0$ is a rest point. In order to plot solution curves note that, if $y \neq 0$,

$$\frac{dy}{dx} = -p\frac{x}{y}$$

and therefore $px + y(dy/dx) = 0$. Along $y = 0$ the slope is infinite. By 3.6 there are solution curves $px^2 + y^2 = c$. As t increases, the point $(x(t), y(t))$ moves in a clockwise direction around an ellipse $px^2 + y^2 = c$ with centre the origin. The value of the constant c may be determined from the position x_0, y_0 when $t = 0$, since then $c = px_0^2 + y_0^2$.

An important special case arises when $p = 1$. Then each ellipse is a circle $x^2 + y^2 = c$.



harmonic motion

A second-order equation which arises more frequently in practice is

$$\frac{d^2x}{dt^2} + q\frac{dx}{dt} + px = 0,$$

where $p > 0$ and $q \neq 0$. It is often called the equation for *damped*

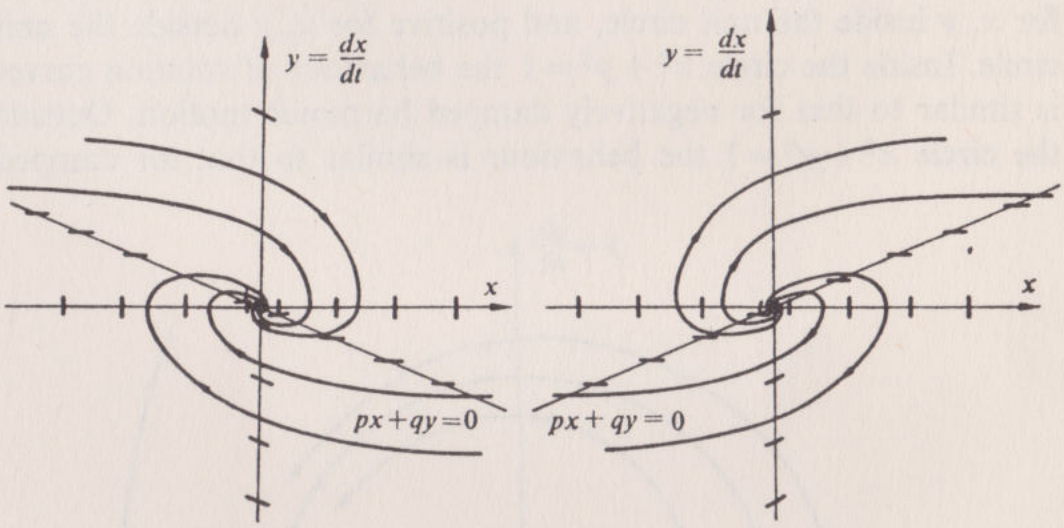
harmonic motion if $q>0$, or *negatively damped harmonic motion* if $q<0$, provided in both cases that q is small. It can be written as an autonomous system

$$\frac{dx}{dt}=y,\ \frac{dy}{dt}=-px-qy.$$

The equations imply that the origin $x=y=0$ is a rest point. As t increases, the point $(x(t), y(t))$ moves in a clockwise direction and since

$$\frac{d}{dt}(px^2+y^2)=2px\frac{dx}{dt}+2y\frac{dy}{dt}=-2q\,y^2$$

it moves inwards or outwards according as $q>0$ or $q<0$. If $q^2<4p$ the resulting curves are spirals, as is proved in 4.5.



$q>0$
damped
harmonic motion

$q^2<4p$

$q<0$
negatively damped
harmonic motion

In all these examples we shall obtain in 4.5 explicit formulae for $x(t)$, $y(t)$ in terms of t. These imply that there is a solution curve through every point x_0, y_0; however, only four such curves have been drawn on each diagram.

Often it is not possible to obtain an explicit formula for solutions of a differential equation, and in such cases the kind of curve

sketching used in the above examples must suffice. For example consider the second-order equation

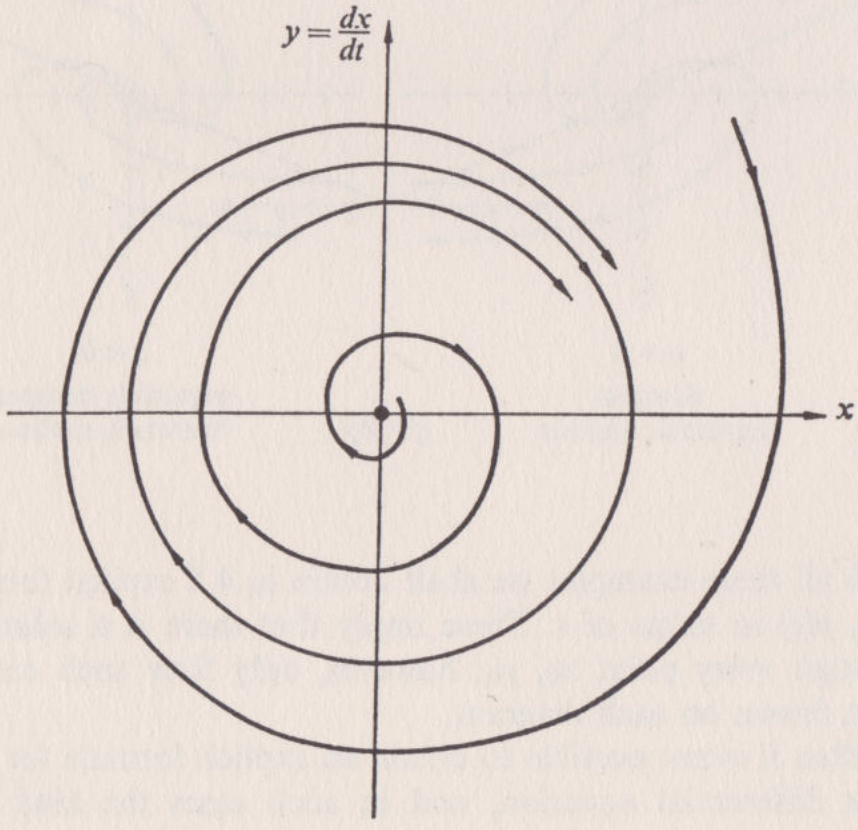
$$\frac{d^2x}{dt^2} - b\left(1 - x^2 - \left(\frac{dx}{dt}\right)^2\right)\frac{dx}{dt} + x = 0$$

where $b > 0$. It can be written as an autonomous system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x + b(1 - x^2 - y^2)y.$$

The equations imply that the origin $x = y = 0$ is a rest point. A comparison with the equation for harmonic motion shows that moreover the unit circle is a solution curve (since if $b(1 - x^2 - y^2) = 0$ the system becomes $dx/dt = y$, $dy/dt = -x$ for which $x^2 + y^2 = c$ is always a solution curve). The term $q = -b(1 - x^2 - y^2)$ is negative for x, y inside the unit circle, and positive for x, y outside the unit circle. Inside the circle $x^2 + y^2 = 1$ the behaviour of solution curves is similar to that for negatively damped harmonic motion. Outside the circle $x^2 + y^2 = 1$ the behaviour is similar to that for damped



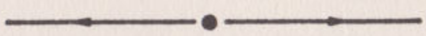
harmonic motion. The solution $x=y=0$ is unstable, since points near (0, 0) move away. The solution curve $x^2+y^2=1$ is stable since nearby points spiral in towards it.

4.4 Linear systems

An autonomous equation of the type considered in 3.3 is said to be *linear* if it has the form

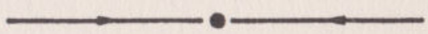
$$\frac{dx}{dt}=rx$$

for some real number r. We should warn that in 3.7 the word linear is used in a slightly different sense, although both have in common the fact that no higher powers of x occur. Note that if r, s are real numbers an equation of the form $dx/dt=rx+s$ can be reduced to the previous form by a change of coordinates which replaces $x+(s/r)$ by x. The results of 3.7 show that every solution of $dx/dt=rx$ has the form $x=x_0e^{rt}$. The origin $x=0$ is a rest point. If $r>0$ the movement is away from the origin; if $r<0$ the movement is towards the origin. Essentially only three distinct situations occur, which are summarized in the diagram.



$r>0$

unstable rest point



$r<0$

stable rest point

$r=0$

line of rest points

An autonomous system in two variables is said to be *linear* if it has the form

$$\frac{dx}{dt}=rx+sy$$
$$\frac{dy}{dt}=-px-qy \qquad (*)$$

where p, q, r, s are real numbers. Note that we no longer assume $y=(dx/dt)$ as in 4.2 and 4.3. This would correspond to choosing

$r=0$, $s=1$, which is in fact the most important special case of (*) because it is equivalent to the second-order equation

$$\frac{d^2x}{dt^2}+q\frac{dx}{dt}+px=0.$$

A solution to (*) will be a pair of functions $t\mapsto(x(t), y(t))$. In accordance with the examples of 4.2 and 4.3, the point $(x(t), y(t))$ will be imagined as moving along a solution curve in the (x, y)-plane as t increases. Our aim is to list the various patterns of solution curves which can arise. For one variable we have just seen there are essentially three distinct situations; in the two-variable case we list fourteen distinct situations in 4.5. The basic method is first to seek straightforward solutions, and then to build up the remaining solutions by taking linear combinations of these. The following lemmas, therefore, are devoted to results on linear combinations of solutions.

Lemma. *Suppose that $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$ are two solutions of (*), and that c_1, c_2 are real numbers. Then*

$$c_1x_1(t)+c_2x_2(t), \qquad c_1y_1(t)+c_2y_2(t)$$

is a solution of (*).

Proof. Let $x=c_1x_1+c_2x_2$ and $y=c_1y_1+c_2y_2$. Then

$$\begin{aligned}\frac{dx}{dt}&=c_1\frac{dx_1}{dt}+c_2\frac{dx_2}{dt}\\&=c_1(rx_1+sy_1)+c_2(rx_2+sy_2)\\&=rx+sy.\end{aligned}$$

Similarly $dy/dt=-px-qy$. Therefore x, y is a solution of (*).

In the next lemma we consider for the first time *complex* solutions of a differential equation. It is often convenient to consider a pair (u_0, v_0) of real numbers as a single complex number $x_0=u_0+iv_0$. Similarly a pair of functions $t\mapsto(u(t), v(t))$ can be regarded as a single complex function $t\mapsto x(t)$, $t\in\mathscr{D}$. Here $\mathscr{D}$ is, as before, a subset of $\mathbf{R}$, and $x(t)$ is a complex number. We write $x=u+iv$ and $dx/dt=du/dt+i(dv/dt)$, and call the functions re $x=u$ and im $x=v$ the 'real' and 'imaginary' parts of x. A complex function

is therefore defined by a description of its real and imaginary parts. For example, if $x = u + iv$ is a complex number we define

$$e^x = e^{u+iv} = e^u(\cos v + i \sin v) = e^u \cos v + ie^u \sin v.$$

There are historical reasons for the names 'real', 'imaginary' and 'complex', but essentially all that is involved is a convenient algebraic mechanism for handling pairs of numbers and pairs of functions, by using the ordinary properties of addition and multiplication together with the requirement that $i^2 = -1$.

A pair of *complex* functions $x(t)$, $y(t)$ is said to be a complex solution of (*) if, with the above definition of derivative,

$$\frac{dx}{dt} = rx + sy$$

$$\frac{dy}{dt} = -px - qy.$$

Since p, q, r, s are real numbers, it is almost immediate that this statement is equivalent to the following: both re $x(t)$, re $y(t)$ and im $x(t)$, im $y(t)$ are solutions of (*). The previous lemma now implies

Lemma. *Suppose that* $x(t)$, $y(t)$ *is a complex solution of* (*), *and that* a, b *are real numbers. Then*

$$a \operatorname{re} x(t) + b \operatorname{im} x(t), \qquad a \operatorname{re} y(t) + b \operatorname{im} y(t)$$

is a solution of (*).

4.5 Classification of linear systems

As in 4.4 we use the notation (*) for the linear autonomous system

$$\frac{dx}{dt} = rx + sy$$

$$\frac{dy}{dt} = -px - qy$$

where p, q, r, s are real numbers. The equations imply that the origin $x = y = 0$ is a rest point. Other solutions $x(t)$, $y(t)$ may be regarded as curves in the (x, y)-plane. The aim is to classify all

patterns which occur, to show how these depend on the numbers p, q, r, s, and finally to obtain explicit formulae for the solutions.

The first step is to ask: is there a point x_1, y_1 for which there is a solution along the line joining x_1, y_1 to the origin? Suppose for a moment that such a point exists. Then the corresponding solution must have the form

$$x = x_1 f(t)$$
$$y = y_1 f(t)$$

where $t \mapsto f(t)$ is a function. Then $dx/dt = x_1 f'(t)$ and $dy/dt = y_1 f'(t)$ so that substitution in (*) gives

$$x_1 f'(t) = (rx_1 + sy_1) f(t),$$
$$y_1 f'(t) = (-px_1 - qy_1) f(t).$$

It follows that $f(t)$ is some constant multiple of e^{mt} where

$$mx_1 = rx_1 + sy_1,$$
$$my_1 = -px_1 - qy_1.$$

Eliminating x_1, y_1 we conclude that m must be a root of the equation

$$m^2 + (q - r)m + (ps - qr) = 0.$$

We call this the *characteristic equation* of the linear system (*).

Conversely, suppose m_1 is a root of the characteristic equation. Except in the case $r - m_1 = s = 0$, any x_1, y_1 such that

$$(r - m_1)x_1 + sy_1 = 0$$

automatically also satisfies the further equation $px_1 + (q + m_1)y_1 = 0$. Therefore there is a solution

$$x = x_1 e^{m_1 t}$$
$$y = y_1 e^{m_1 t}$$

of (*). If m_1, m_2 are two distinct roots of the characteristic equation, then there is another solution

$$x = x_2 e^{m_2 t}$$
$$y = y_2 e^{m_2 t}$$

of (*). Applying the first lemma of 4.4 we have

Theorem I. *Suppose that the characteristic equation has two distinct roots m_1, m_2. Let (x_1, y_1) and (x_2, y_2) be non-zero points such that*

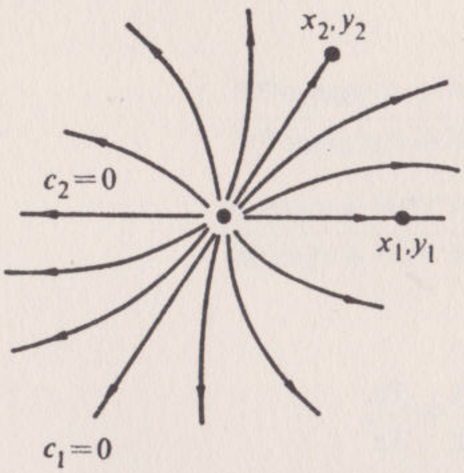
$$(r - m_1)x_1 + sy_1 = px_1 + (q + m_1)y_1 = 0$$
$$(r - m_2)x_2 + sy_2 = px_2 + (q + m_2)y_2 = 0.$$

There are solutions

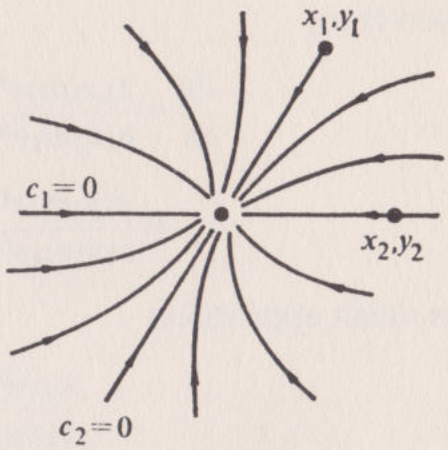
$$x = c_1 x_1 e^{m_1 t} + c_2 x_2 e^{m_2 t}$$
$$y = c_1 y_1 e^{m_1 t} + c_2 y_2 e^{m_2 t}$$

of (*), *where* c_1, c_2 *are arbitrary real numbers.*

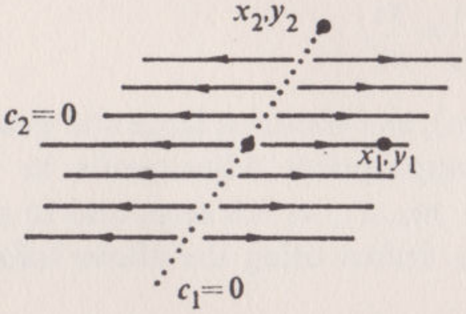
I(i) $m_1 > m_2 > 0$
unstable node



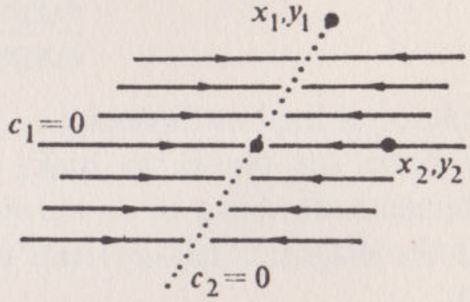
I(ii) $0 > m_1 > m_2$
stable node



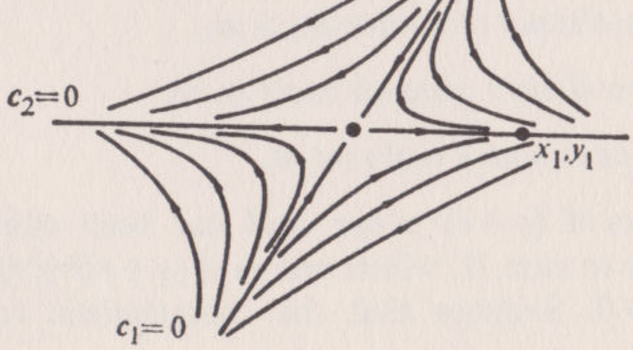
I(iii) $m_1 > m_2 = 0$
unstable line of rest points



I(iv) $0 = m_1 > m_2$
stable line of rest points



I(v) $m_1 > 0 > m_2$
saddle point



Remark. Since $m_1 \neq m_2$ the points (x_1, y_1) and (x_2, y_2) lie on distinct lines through the origin. Therefore given x_0, y_0 it is possible to find a unique solution of the above form such that $x = x_0$ and $y = y_0$ when $t = 0$. In fact it can be proved that if $m_1 \neq m_2$ every solution to (*) is of the above form. The curve patterns in the (x, y)-plane corresponding to the above solutions are of five basic types I(i)–I(v). Note that if $m_1 > m_2$, and if c_1, c_2, m_1, m_2, x_1, x_2 are all non-zero,

$$\frac{dy}{dx} = \frac{c_1 y_1 m_1 e^{m_1 t} + c_2 y_2 m_2 e^{m_2 t}}{c_1 x_1 m_1 e^{m_1 t} + c_2 x_2 m_2 e^{m_2 t}}$$

$$= \frac{c_1 y_1 m_1 e^{(m_1 - m_2)t} + c_2 y_2 m_2}{c_1 x_1 m_1 e^{(m_1 - m_2)t} + c_2 x_2 m_2}.$$

Thus dy/dx approaches

$$\frac{c_2 y_2 m_2}{c_2 x_2 m_2} = \frac{y_2}{x_2},$$

the slope of the line through (x_2, y_2), as t becomes large and negative. Similarly dy/dx approaches

$$\frac{c_1 y_1 m_1}{c_1 x_1 m_1} = \frac{y_1}{x_1},$$

the slope of the line through (x_1, y_1), as t becomes large and positive. We leave the reader to make appropriate adjustments to these statements when any of c_1, c_2, m_1, m_2, x_1, x_2 are zero, and to check that the diagrams I(i)–I(v) can be drawn using the above information.

In order to complete the classification of solutions of the linear system (*) it is necessary to consider the possibility that the characteristic equation does not have two distinct real roots. In all there are three cases.

Case I: *two distinct real roots* $m_1 > m_2$.

Case II: *two distinct complex roots* m, $\bar{m}$.

Case III: *one repeated real root* m.

Case I occurs if $(q + r)^2 > 4ps$, and has been considered above. We now turn to case II, which occurs if $(q + r)^2 < 4ps$. In this case therefore $s \neq 0$. Suppose that the characteristic equation has a

complex root $m = f + ig$, where $g \neq 0$. Then the other root must be $\bar{m} = f - ig$. Let $k = (m - r)/s$. Then the complex number k can be written in terms of real and imaginary parts, say $k = u + iv$. (In fact $u = (f - r)/s$ and $v = g/s$.)

By the same argument as was used in the case when m is real, there is a complex solution of (*)

$$x = e^{mt}$$
$$y = ke^{mt}$$

which can be separated into real and imaginary parts

$$x = e^{ft} \cos gt + i\, e^{ft} \sin gt$$
$$y = e^{ft} (u \cos gt - v \sin gt) + ie^{ft} (v \cos gt + u \sin gt).$$

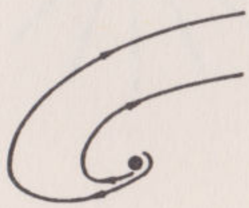
Applying the second lemma of 4.4 we have

Theorem II. *Suppose that the characteristic equation has two distinct complex roots $m, \bar{m}$. Let $m = f + ig$ and $k = (m - r)/s = u + iv$. There are solutions*

$$x = ae^{ft} \cos gt + be^{ft} \sin gt$$
$$y = ae^{ft} (u \cos gt - v \sin gt) + be^{ft} (v \cos gt + u \sin gt)$$

of (*), *where a, b are arbitrary real numbers.*

II(i) $f > 0$
unstable focus



II(ii) $f < 0$
stable focus

II(iii) $f = 0$
centre or vortex point

Remark. Since $m \neq \bar{m}$ we have $v \neq 0$. Therefore given x_0, y_0 it is possible to find a unique solution of the above form such that $x = x_0$ and $y = y_0$ when $t = 0$. In fact it can be proved that if $m \neq \bar{m}$ every solution to (*) is of the above form. The curve patterns in the (x, y)-plane corresponding to the above solutions are of three basic types II(i)–II(iii) illustrated in the diagram. The flow may be in a clockwise or anticlockwise direction; the diagram is drawn with flow clockwise.

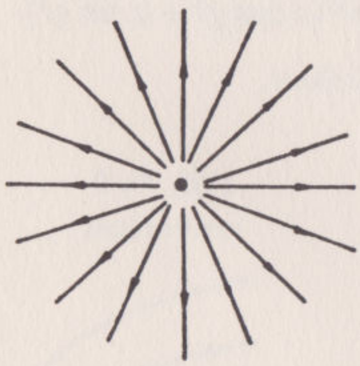
It remains to consider case III, in which the characteristic equation has a repeated root m. Then $(q+r)^2 = 4ps$ and $m = \frac{1}{2}(r-q)$, so that also $(r-m)^2 = (q+m)^2 = ps$.

One possibility is that $p = s = 0$ and $m = r = -q$. In this case (*) becomes

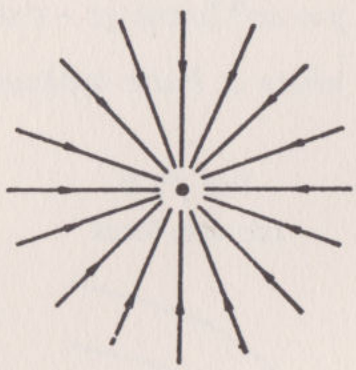
$$\frac{dx}{dt} = mx$$

$$\frac{dy}{dt} = my.$$

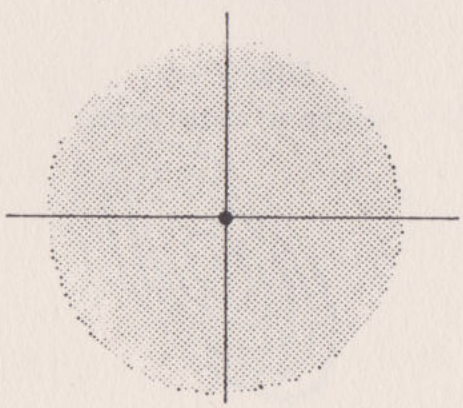
III(i) $m > 0$
unstable node



III(ii) $m < 0$
stable node



III(iii) $m = 0$
plane of rest points



Then (x_0e^{mt}, y_0e^{mt}) is a solution of (*) for any real numbers x_0, y_0. The corresponding curve patterns III(i), (ii) are similar to I(i), (ii) but every curve is straight. Again, there is a unique curve through each point (x_0, y_0). In case III (iii) the only solutions are constants $x = x_0$, $y = y_0$.

The other possibility is that p, s are not both zero. In this case the repeated root m yields solutions

$$x = x_1e^{mt}$$

$$y = y_1e^{mt}$$

where the point (x_1, y_1) is determined up to a scalar multiple by the equation

$$(r-m)x_1 + sy_1 = 0,$$

or the essentially equivalent equation

$$-px_1 - (q+m)y_1 = 0.$$

Now choose some point (x_2, y_2) which is not a multiple of (x_1, y_1), and define

$$a_1 = (r-m)x_2 + sy_2,$$

$$b_1 = -px_2 - (q+m)y_2.$$

It follows from the relations $(r-m)^2 = (q+m)^2 = ps$ and $2m = r - q$ that a_1, b_1 satisfy $(r-m)a_1 + sb_1 = 0$ and $-pa_1 - (q+m)b_1 = 0$. Therefore (a_1, b_1) is some multiple of (x_1, y_1). Thus there is a real number f such that $a_1 = fx_1$ and $b_1 = fy_1$. Consider the functions

$$x = (fx_1t + x_2)e^{mt},$$

$$y = (fy_1t + y_2)e^{mt}.$$

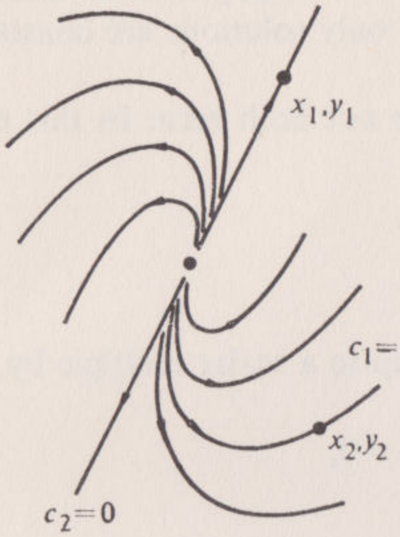
It is easy to check that the above relations imply that x, y is another solution of (*). By the first lemma of 4.4 any linear combination of the two solutions obtained is also a solution. Therefore if c_1, c_2 are arbitrary real numbers the functions

$$x = (c_1 + c_2ft)x_1e^{mt} + c_2x_2e^{mt}$$

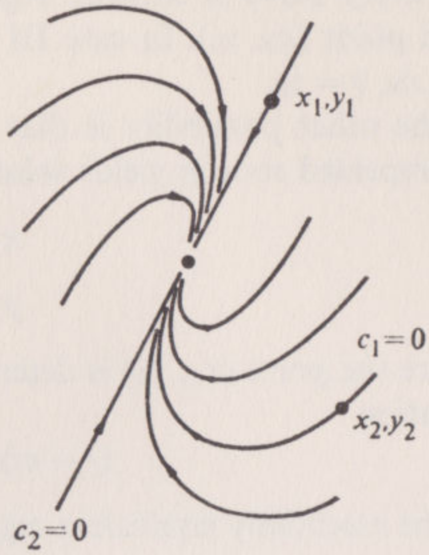
$$y = (c_1 + c_2ft)y_1e^{mt} + c_2y_2e^{mt}$$

give a solution of (*). Again there is a unique curve through each point of the plane. The curve patterns III(iv), (v) are similar to

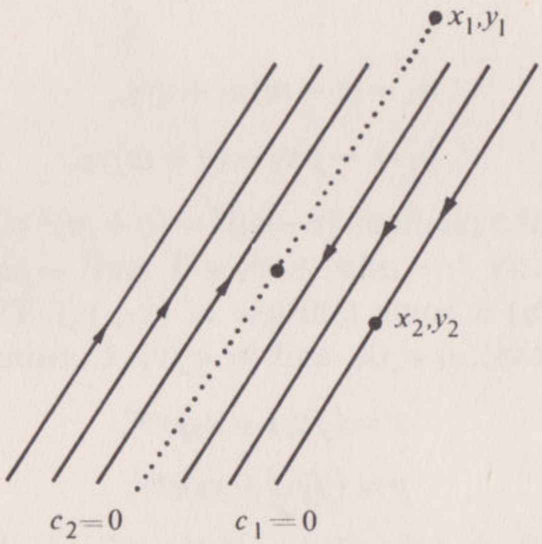
III(iv) $m > 0$
unstable node



III(v) $m < 0$
stable node



III(vi) $m = 0$
boundary layer



I(i), (ii) but now the two straight line solutions have coincided, and dy/dx approaches the slope of the line through (x_1, y_1) both as t becomes large positive, and as t becomes large negative. For III(vi) see example 2.12: points on opposite sides of the line of rest points move in opposite directions.

Remarks. Not all the fourteen types of linear system I (i)–(v), II(i)–(iii), III(i)–(vi) are of the same significance from a physical

point of view. In most physical situations we expect structural stability: that is, a small change in the coefficients p, q, r, s leads to a small change in the solution curves.

Clearly a small change in the equation giving I(iii) will yield either situation I(i) or I(v). Therefore I(iii) is not structurally stable; neither are I(iv), II(iii) or III(i)–(vi). However, the fact that a situation is unlikely to occur in physics does not prevent its being of interest to the mathematician (and even to the physicist – as is shown by the example of harmonic motion, of type II(iii), and by the example of zero acceleration, of type III(vi)).

4.6 Second-order linear equations

The detailed results of 4.5 can be applied to the second-order equation

$$\frac{d^2x}{dt^2}+q\frac{dx}{dt}+px=0, \qquad (**)$$

to find solutions $t\mapsto x(t)$. Here p, q are real numbers and the corresponding autonomous system is

$$\frac{dx}{dt}=y,$$

$$\frac{dy}{dt}=-px-qy.$$

In the analysis of 4.5 we put $r=0$, $s=1$. Then the characteristic equation of the system becomes

$$m^2+qm+p=0.$$

This equation is also called the *auxiliary equation* of (**). If m is a root of the auxiliary equation we take $x_1=1$, $y_1=m-r$ as the non-zero point satisfying $(r-m)x_1+sy_1=0$. Then Theorem I of 4.5 implies immediately:

Theorem I. *If* $q^2>4p$ *the auxiliary equation of* (**) *has two distinct real roots* $m_1=-\tfrac{1}{2}q+\sqrt{(q^2-4p)}$ *and* $m_2=-\tfrac{1}{2}q-\sqrt{(q^2-4p)}$. *There are solutions*

$$x=c_1e^{m_1t}+c_2e^{m_2t}$$

of (**), *where* c_1, c_2 *are arbitrary real numbers.*

The corresponding diagram of solution curves in phase space will be of

type I(i), unstable node, if $q^2 > 4p > 0$, $q < 0$,

type I(ii), stable node, if $q^2 > 4p > 0$, $q > 0$,

type I(iii), unstable line of rest points, if $p = 0$, $q < 0$,

type I(iv), stable line of rest points, if $p = 0$, $q > 0$,

type I(v), saddle point, if $p < 0$.

In the same way Theorem II of 4.5 implies

Theorem II. *If* $q^2 < 4p$ *the auxiliary equation of* (**) *has two distinct complex roots* $f \pm ig$, *where* $f = -\frac{1}{2}q$ *and* $g = \frac{1}{2}\sqrt{(4p - q^2)}$. *There are solutions*

$$x = ae^{ft}\cos gt + be^{ft}\sin gt$$

of (**), *where* a, b *are arbitrary real numbers.*

The corresponding diagram of solution curves in phase space will be of

type II(i), unstable focus, if $q^2 < 4p$, $q < 0$,

type II(ii), stable focus, if $q^2 < 4p$, $q > 0$,

type II(iii), centre, if $p > 0$, $q = 0$.

The final case is dealt with by

Theorem III. *If* $q^2 = 4p$ *the auxiliary equation of* (**) *has a repeated root* $m = -\frac{1}{2}q$. *There are solutions*

$$x = (c_1 + c_2 t)e^{mt}$$

of (**), *where* c_1, c_2 *are arbitrary real numbers.*

Proof. We put $r = 0$, $s = 1$ in the analysis of 4.5. Since $s \neq 0$ the cases III(i), III(ii), III(iii) cannot arise. We choose $x_1 = 1$, $y_1 = -\frac{1}{2}q$ and $x_2 = 0$, $y_2 = 1$. It then follows that $a_1 = 1$, $b_1 = -\frac{1}{2}q$ and hence that $f = 1$. Therefore the final solution obtained in 4.5 becomes $x = (c_1 + c_2 t)e^{mt}$.

The corresponding diagram of solution curves in phase space will be of

type III(iv), unstable node, if $q^2 = 4p$, $q < 0$

type III(v), stable node, if $q^2 = 4p$, $q > 0$

type III(vi), boundary layer, if $p = q = 0$.

4.7. Examples

(i) Consider the equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0.$$

The auxiliary equation $m^2 - 3m + 2 = 0$ has roots $m = 1$ and $m = 2$. By Theorem I there are solutions

$$x = c_1e^t + c_2e^{2t}$$

where c_1, c_2 are real numbers.

(ii) Consider the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0.$$

The auxiliary equation $m^2 + m + 1 = 0$ has roots $m = -\frac{1}{2} \pm i(\sqrt{3}/2)$. By Theorem II there are solutions

$$x = ae^{-\frac{1}{2}t}\cos\left(t\frac{\sqrt{3}}{2}\right) + be^{-\frac{1}{2}t}\sin\left(t\frac{\sqrt{3}}{2}\right)$$

where a, b are real numbers.

(iii) Consider the equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0.$$

The auxiliary equation $m^2 - 2m + 1 = 0$ has a repeated root $m = 1$. By Theorem III there are solutions

$$x = (c_1 + c_2t)e^t$$

where c_1, c_2 are real numbers.

Exercises

1 Let

$$\frac{dx}{dt} = -\lambda x$$

be the equation controlling the radioactive decay of an element X, as in 4.1. The *half life* T of X is defined as the time taken for half a given amount of X to decay. Prove that

$$T = \frac{1}{\lambda}\log 2.$$

2 Consider the autonomous system

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = x + 2y.$$

Show that there are solution curves lying along the lines $x = 0$ and $x + y = 0$. Deduce that the solution curve through (x_0, y_0) is

$$x = x_0 e^t$$

$$y = (x_0 + y_0)e^{2t} - x_0 e^t.$$

3 Use the substitution $t = e^u$ to solve the differential equation

$$t^2 \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x = 0, \qquad t > 0.$$

What is the appropriate substitution in the region $t < 0$?

4 Write down differential equations with the following solutions:

(i) $x = e^t (a \cos (t\sqrt{3})\, t + b \sin (t\sqrt{3}))$,
(ii) $x = ae^t + be^{-2t}$,
(iii) $x = (a + bt)e^{-t}$.

5 Find all solutions of each of the following differential equations.

(i) $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 0$, (ii) $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 0$,

(iii) $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 5x = 0$, (iv) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} = 0$.

6 Find a solution of each of the following differential equations with the property: $x = 0$ and $dx/dt = 1$ when $t = 0$.

(i) $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$,

(ii) $\dfrac{d^2x}{dt^2} + x = 0$,

(iii) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0$,

(iv) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + 2x = 0$.

CHAPTER FIVE

Linear equations

The word 'linear' has already been used in three senses: to describe first-order equations of the form $(dx/dt)+xg(t)=h(t)$ in 3.7, to describe the autonomous system

$$\frac{dx}{dt}=rx+sy$$

$$\frac{dy}{dt}=-px-qy$$

in 4.5, and to describe the second-order equation

$$\frac{d^2x}{dt^2}+q\,\frac{dx}{dt}+px=0$$

in 4.6. In this chapter we seek to justify our use of the word by placing these three examples in a single framework. The essential feature is that given two solutions of the equation, any linear combination of them will be a solution either of the given equation, or of a closely related equation. This fact leads directly to the results often summarized as rules about 'arbitrary constants', 'complementary functions', and 'particular integrals'.

5.1 Linear operators

Let $\mathscr{D}$ be a subset of $\mathbf{R}$ which is kept fixed in all that follows, but which in practice would often be the whole of $\mathbf{R}$. We consider functions $t\mapsto x(t)$ with domain $\mathscr{D}$. The idea behind linearity is that

any linear combination of functions with domain $\mathscr{D}$ is itself a function with domain $\mathscr{D}$. If c_1, c_2 are real numbers and x_1, x_2 are functions with domain $\mathscr{D}$ then $c_1x_1 + c_2x_2$ is the function defined for all $t \in \mathscr{D}$ by

$$(c_1x_1 + c_2x_2)(t) = c_1x_1(t) + c_2x_2(t).$$

Let $\mathbf{V}$ be the collection of all functions $t \mapsto x(t)$ with domain $\mathscr{D}$ which are infinitely differentiable; that is, the r-th derivative d^rx/dt^r exists for every integer $r > 0$. If x_1, x_2 are functions in $\mathbf{V}$ then so is $c_1x_1 + c_2x_2$ because

$$\frac{d^r}{dt^r}(c_1x_1 + c_2x_2) = c_1 \frac{d^rx_1}{dt^r} + c_2 \frac{d^rx_2}{dt^r}.$$

It is often convenient psychologically to think of x as an element, or point, of $\mathbf{V}$ rather than as a function $t \mapsto x(t)$. This makes it possible to consider functions which associate, to each element $x \in \mathbf{V}$, an element $T(x) \in \mathbf{V}$. We use the notation $T: \mathbf{V} \to \mathbf{V}$ for such a function, but give it a special name to avoid confusion with the functions $t \mapsto x(t)$ which are elements of $\mathbf{V}$.

Definition. A function $T: \mathbf{V} \to \mathbf{V}$ is a *linear operator* if, for all $c_1, c_2 \in \mathbf{R}$ and $x_1, x_2 \in \mathbf{V}$,

$$T(c_1x_1 + c_2x_2) = c_1T(x_1) + c_2T(x_2).$$

Examples. (i) Let c be a real number. The function $x \mapsto cx$ is a linear operator $c: \mathbf{V} \to \mathbf{V}$.

(ii) Differentiation $x \mapsto (dx/dt)$ defines a linear operator $D: \mathbf{V} \to \mathbf{V}$. Similarly the r-th derivative $x \mapsto (d^rx/dt^r)$ defines a linear operator $D^r: \mathbf{V} \to \mathbf{V}$.

(iii) If $g \in \mathbf{V}$ the $x \mapsto (dx/dt) + xg$ is a linear operator $(D + g): \mathbf{V} \to \mathbf{V}$.

(iv) If p, q are real numbers then

$$x \mapsto \frac{d^2x}{dt^2} + q\frac{dx}{dt} + px$$

is a linear operator $(D^2 + qD + p): \mathbf{V} \to \mathbf{V}$.

(v) If $a_1, \ldots, a_n \in \mathbf{V}$ then

$$x \mapsto \frac{d^nx}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_nx$$

is a linear operator $(D^n + a_1 D^{n-1} + \cdots + a_n): \mathbf{V} \to \mathbf{V}$. Here $a_1, \ldots, a_n$ are elements of $\mathbf{V}$ (i.e. functions with domain $\mathscr{D}$). If $a_1, \ldots, a_n$ are all real numbers (i.e. constant functions) the linear operator is said to be of n-th order with constant coefficients.

The above examples suggest that certain differential equations can be written in the form $T(x) = 0$, where $T: \mathbf{V} \to \mathbf{V}$ is a linear operator. This leads to the following terminology. If $T: \mathbf{V} \to \mathbf{V}$ is a linear operator, the collection $\mathbf{V}_0$ of all $x \in \mathbf{V}$ such that $T(x) = 0$ is called the *solution set*, *solution space* or *kernel* of T. An element $x \in \mathbf{V}_0$ is called a *solution* of the equation $T(x) = 0$.

The definition of linear operator implies immediately:

Lemma. *Let $T: \mathbf{V} \to \mathbf{V}$ be a linear operator and $\mathbf{V}_0 = \{x \in \mathbf{V} : T(x) = 0\}$. If $x_1, x_2 \in \mathbf{V}_0$ then $c_1 x_1 + c_2 x_2$ is an element of $\mathbf{V}_0$ for any real numbers c_1, c_2.*

Translated in terms of solutions this lemma becomes: if x_1, x_2 are two solutions of the equation $T(x) = 0$ then $c_1 x_1 + c_2 x_2$ is a solution for any real numbers c_1, c_2.

This may be compared with the first lemma of 4.4 which results from the fact that the function

$$(x, y) \mapsto \left(\frac{dx}{dt} - rx - sy, \frac{dy}{dt} + px + qy\right)$$

is a linear operator $\mathbf{V}^2 \to \mathbf{V}^2$. Here $\mathbf{V}^2$ denotes the set of pairs (x, y) with $x \in \mathbf{V}$, $y \in \mathbf{V}$, and linear combination of pairs defined by

$$c_1(x_1, y_1) + c_2(x_2, y_2) = (c_1 x_1 + c_2 x_2, c_1 y_1 + c_2 y_2).$$

This explains the use of the word *linear* in 4.5.

More generally the words *linear equation* are often applied to an equation of the form

$$T(x) = h$$

where $T: \mathbf{V} \to \mathbf{V}$ is a linear operator, and h is a fixed element of $\mathbf{V}$ (i.e. a function $t \mapsto h(t)$ with domain $\mathscr{D}$), for example the equation

$$\frac{dx}{dt} + xg(t) = h(t)$$

considered in 3.7 is of this kind, with T the linear operator $D + g$ of example (iii) above.

Lemma. *Let $T: \mathbf{V} \to \mathbf{V}$ be a linear operator, and $y \in \mathbf{V}$ a particular solution of the equation $T(x) = h$. Then every solution of $T(x) = h$ has the form $y_0 + y$, where y_0 is a solution of the equation $T(x) = 0$.*

Proof. The definition of linear operator implies that

$$T(y_0 + y) = T(y_0) + T(y) = 0 + h = h$$

and therefore $y_0 + y$ is a solution of $T(x) = h$. Similarly if z is an arbitrary solution of $T(x) = h$ then

$$T(z - y) = T(z) - T(y) = h - h = 0$$

and therefore $z - y = y_0$ for some solution y_0 of $T(x) = 0$; in other words, every solution z of $T(x) = h$ has the form $y_0 + y$.

Corollary. *Let $x_1, \ldots, x_n \in \mathbf{V}$ be functions with the property: every solution y_0 of $T(x) = 0$ has the form $y_0 = c_1x_1 + \cdots + c_nx_n$ where $c_1, \ldots, c_n$ are real numbers. If $y \in \mathbf{V}$ is a particular solution of $T(x) = h$ then every solution of $T(x) = h$ has the form*

$$c_1x_1 + \cdots + c_nx_n + y.$$

There are traditional names for some of the terms which occur in the corollary. The function y is called a *particular integral* of $T(x) = h$. The expression $c_1x_1 + \cdots + c_nx_n$ is called the *complementary function*, and the real numbers $c_1, \ldots, c_n$ are called *arbitrary constants*.

Example. Consider the equation

$$(1 + t^2)\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} = 2t.$$

It is of the form $T(x) = h$ where h is the function $t \mapsto 2t$ and T is the linear operator $(1 + t^2)D^2 + 2tD$. The equation can be written

$$\frac{d}{dt}\left((1 + t^2)\frac{dx}{dt}\right) = 2t.$$

Therefore

$$(1 + t^2)\frac{dx}{dt} = 1 + t^2 + c_1$$

where c_1 is a real number. It follows that

$$\frac{dx}{dt} = 1 + \frac{c_1}{1+t^2}$$

$$x = t + c_1 \tan^{-1} t + c_2$$

where c_1, c_2 are real numbers. Here t is the particular integral, and $c_1 \tan^{-1} t + c_2$ is the complementary function. The arbitrary constants c_1, c_2 can be determined as soon as additional information is specified about the desired solution. For instance, suppose we seek a solution to the above equation for which $x = 0$ and $dx/dt = 2$ when $t = 0$. Substitution of these values gives $c_1 = 1$, $c_2 = 0$. Therefore the solution with the required values when $t = 0$ is

$$x = t + \tan^{-1} t.$$

5.2 Linear equations with constant coefficients

Consider the equation

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n x = h(t)$$

where h is a fixed element of $\mathbf{V}$ (i.e. a function $t \mapsto h(t)$ with domain $\mathscr{D}$) and $a_1, \ldots, a_n$ are real numbers. It can be written in the form

$$T(x) = h$$

where $T: \mathbf{V} \to \mathbf{V}$ is the linear operator $D^n + a_1 D^{n-1} + \cdots + a_n$ of example (v) in 5.1. By the second lemma of 5.1 it is possible to find all solutions of the equation $T(x) = h$ in two steps:

(i) find all solutions of $T(x) = 0$, and

(ii) find a particular solution of $T(x) = h$.

A method for carrying out step (ii) is given in 5.4; however, in practice it is often possible to guess a particular solution. We therefore concentrate first on step (i).

As in 4.6 we introduce the *auxiliary equation*

$$m^n + a_1 m^{n-1} + \cdots + a_n = 0.$$

Lemma. *If the real number m satisfies the auxiliary equation $m^n + a_1 m^{n-1} + \cdots + a_n = 0$ then*

$$x = ce^{mt}$$

is a solution of the equation $T(x) = 0$ for any real number c.

Proof. If $x = ce^{mt}$

$$\begin{aligned} T(x) &= (D^n + a_1 D^{n-1} + \cdots + a_n) ce^{mt} \\ &= D^n(ce^{mt}) + a_1 D^{n-1}(ce^{mt}) + \cdots + a_n(ce^{mt}) \\ &= m^n(ce^{mt}) + a_1 m^{n-1}(ce^{mt}) + \cdots + a_n(ce^{mt}) \\ &= (m^n + a_1 m^{n-1} + \cdots + a_n) ce^{mt} = 0. \end{aligned}$$

It may happen that the auxiliary equation has a complex root $m = f + ig$. In this case the same argument will imply that any multiple of

$$e^{mt} = e^{ft+igt} = e^{ft}(\cos gt + i \sin gt)$$

is a *complex* solution of the equation $T(x) = 0$. The same argument used already in the second lemma of 4.4 then implies:

Lemma. *If the complex number $m = f + ig$ satisfies the auxiliary equation $m^n + a_1 m^{n-1} + \cdots + a_n = 0$ then*

$$x = ae^{ft} \cos gt + be^{ft} \sin gt$$

is a solution of the equation $T(x) = 0$ for any real numbers a, b.

By taking linear combinations of the solutions given by the preceding lemmas, we can build up more complicated solutions of the equation $T(x) = 0$. The result is:

Theorem. *Let $T: \mathbf{V} \to \mathbf{V}$ be the linear operator*

$$D^n + a_1 D^{n-1} + \cdots + a_n$$

where $a_1, \ldots, a_n$ are real numbers. Suppose that the auxiliary equation

$$m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

has n distinct roots: $2s$ complex roots $f_1 \pm ig_1, \ldots, f_s \pm ig_s$ and $n - 2s$ real roots $m_1, \ldots, m_{n-2s}$. Consider the functions

$$\begin{aligned} x_j &= e^{f_j t} \cos g_j t & 1 \leqslant j \leqslant s \\ x_{s+j} &= e^{f_j t} \sin g_j t & 1 \leqslant j \leqslant s \\ x_{2s+k} &= e^{m_k t} & 1 \leqslant k \leqslant n - 2s. \end{aligned}$$

Then for any real numbers $c_1, \ldots, c_n$ the equation $T(x)=0$ has a solution

$$x = c_1x_1 + \cdots + c_nx_n.$$

This theorem contains as a special case the solution to the equation $(dx/dt)+xa=0$ obtained in 3.7, and also cases I and II of the solutions to the equation $(d^2x/dt^2)+q(dx/dt)+px=0$ obtained in 4.6. To obtain a generalization of case III we must allow for the possibility that the auxiliary equation has some repeated roots.

It can be checked explicitly that, if m is a root of the auxiliary equation of multiplicity r, the functions t^ke^{mt} are solutions of the equation $T(x)=0$ for $k=0, 1, \ldots, r-1$. This implies:

Proposition. *Suppose that in the above theorem $x_i = \cdots = x_{i+r-1}$ because of a root m of multiplicity r. Then in the expression $x=c_1x_1+\cdots+c_nx_n$ the terms $c_ix_i+\cdots+c_{i+r-1}x_{i+r-1}$ may be replaced by $(c_i+c_{i+1}t+\cdots+c_{i+r-1}t^{r-1})x_i$.*

Remark. It is also true, although we have not given the proof, that every solution of the equation $T(x)=0$ is of the form specified in the theorem when the auxiliary equation has distinct roots. If there are some repeated roots then again every solution is of the form specified, provided that the expression $c_1x_1+\cdots+c_nx_n$ is modified as in the above proposition.

Examples. (i) Consider the equation

$$\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 11\frac{dx}{dt} - 6x = 20 \cos t.$$

This has the form $T(x)=h$ where $T=D^3-6D^2+11D-6$ and h is the function $t \mapsto 20 \cos t$. The auxiliary equation is

$$0 = m^3 - 6m^2 + 11m - 6 = (m-1)(m-2)(m-3).$$

Therefore every solution of $T(x)=0$ has the form

$$x = c_1e^t + c_2e^{2t} + c_3e^{3t}.$$

This is the complementary function. To find a particular solution it is reasonable, since $h(t)=20 \cos t$, to try out linear combinations of $\cos t$ and $\sin t$. In fact

$$T(\cos t) = \sin t + 6\cos t - 11 \sin t - 6 \cos t = -10 \sin t,$$
$$T(\sin t) = -\cos t + 6 \sin t + 11 \cos t - 6 \sin t = 10 \cos t.$$

Therefore $x = 2 \sin t$ is a particular solution, and every solution is of the form

$$x = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} + 2 \sin t.$$

In practice the arbitrary constants c_1, c_2, c_3 are often determined by the *initial conditions*, i.e. the values of x and its derivatives for some fixed value of t. For instance the solution to the above equation may be required which has

$$x = 0, \qquad \frac{dx}{dt} = 0, \qquad \frac{d^2x}{dt^2} = -6$$

when $t = 0$. Substitution of these values gives

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= c_1 + 2c_2 + 3c_3 + 2 \\ -6 &= c_1 + 4c_2 + 9c_3 \end{aligned}$$

and hence $c_1 = 2$, $c_2 = -2$, $c_3 = 0$. Therefore the required solution is

$$x = 2(e^t - e^{2t} + \sin t).$$

(ii) Consider the equation

$$\frac{d^3x}{dt^3} - 2\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 4e^t.$$

Here $T = D^3 - 2D^2 + D - 2$ and h is the function $t \mapsto 4e^t$. The auxiliary equation is $0 = m^3 - 2m^2 + m - 2 = (m-2)(m^2+1)$. Therefore the complementary function is

$$x = c_1 e^{2t} + c_2 \cos t + c_3 \sin t.$$

To find a particular integral it is reasonable, since $h(t) = 4e^t$, to try out multiples of e^t. In fact

$$T(e^t) = e^t - 2e^t + e^t - 2e^t = -2e^t$$

and therefore $x = -2e^t$ is a particular solution. Thus every solution is of the form

$$x = c_1 e^{2t} + c_2 \cos t + c_3 \sin t - 2e^t.$$

where c_1, c_2, c_3 are real numbers which may be determined from

the initial conditions. For instance the solution with values $x=0$, $dx/dt=0$, $d^2x/dt^2=1$ when $t=0$ has

$$0=c_1+c_2-2$$
$$0=2c_1+c_3-2$$
$$1=4c_1-c_2-2$$

and therefore $c_1=1$, $c_2=1$, $c_3=0$.

(iii) Consider the equation

$$\frac{d^3x}{dt^3}-\frac{d^2x}{dt^2}-\frac{dx}{dt}+x=2.$$

Here $T=D^3-D^2-D+1$ and $h=2$. The auxiliary equation is $0=m^3-m^2-m+1=(m+1)(m-1)^2$. There are solutions of $T(x)=0$ of the form

$$x=c_1e^{-t}+c_2e^t+c_3e^t.$$

However, according to the above proposition, the last two terms may be replaced to obtain the more general expression

$$x=c_1e^{-t}+(c_2+c_3t)e^t.$$

Clearly $x=2$ is a particular solution. Therefore every solution is of the form

$$x=c_1e^{-t}+c_2e^t+c_3te^t+2,$$

where c_1, c_2, c_3 are real numbers which may be determined from the initial conditions.

5.3 Stability

Consider the linear equation

$$T(x)=h$$

where $T=D^n+a_1D^{n-1}+\cdots+a_n$ is as in 5.2. If $y\in\mathbf{V}$ is a particular solution then by the results of 5.1 and 5.2 every other solution has the form

$$x=c_1x_1+\cdots+c_nx_n+y$$

where $c_1, \ldots, c_n$ are real numbers and $x_1, \ldots, x_n$ are functions determined by the auxiliary equation.

Under what circumstances will any two particular solutions become arbitrarily close for large values of t? Since any two particular solutions differ by a function of the form $c_1x_1 + \cdots + c_nx_n$ it is equivalent to ask: for what equations do the functions $x_1, \ldots, x_n$ become arbitrarily small for large values of t?

Definition. The equation $T(x) = 0$ is *stable* if every solution tends to zero as t tends to infinity.

To make this definition precise needs a discussion of limits and continuity. For the present purposes it is sufficient to know the following fact about exponential functions (compare Moss–Roberts, Theorem 4.5.9): if $k \geqslant 0$ then, as t tends to infinity, $t^k e^{mt}$ tends to zero if $m < 0$ and to infinity if $m > 0$.

Theorem. *The equation*

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n x = 0$$

is stable if and only if every root of the auxiliary equation

$$m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

has negative real part.

Proof. By the theorem of 5.2 the roots of the auxiliary equation determine functions $x_1, \ldots, x_n$. The equation is stable if each x_j tends to zero as t tends to infinity. But x_j has one of the forms $t^k e^{m_j t}$, $t^k e^{f_j t} \cos g_j t$, $t^k e^{f_j t} \sin g_j t$, the power of t being zero unless there are repeated roots of the auxiliary equation. In each case the function tends to zero, as t tends to infinity, if and only if $m_j < 0$ or $f_j < 0$.

In the case $n = 2$ this notion of *stable* links up with the terminology already used in 4.5 and 4.6. The statement that

$$\frac{d^2x}{dt^2} + q\frac{dx}{dt} + px = 0$$

is stable corresponds to the statement that the autonomous system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -px - qy$$

has a stable node or focus at $x=y=0$; that is, solution curves run in towards the origin as t tends to infinity. The analysis of 4.6 shows that the equation $(d^2x/dt^2)+q(dx/dt)+px=0$ is stable if and only if $p>0$, $q>0$; this condition agrees with that of the above theorem because the roots of $m^2+qm+p=0$ have real part $-\frac{1}{2}q$ if $q^2<4p$, and real part $-\frac{1}{2}q\pm\frac{1}{2}\sqrt{(q^2-4p)}$ if $q^2>4p$.

The importance of stable equations lies in the fact that often a solution of

$$T(x)=h$$

is required only once a 'steady state' has been reached; in other words only for large values of t. If $T(x)=0$ is stable then any two particular solutions of $T(x)=h$ will become arbitrarily close for large values of t, so that for many purposes it does not matter which particular solution is chosen: the behaviour of the solution for large values of t is independent of the initial conditions.

For example consider the equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+3x=9.$$

Since $2>0$ and $3>0$ the equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+3x=0$$

is stable. Moreover $x=3$ is clearly a particular solution. Therefore every solution tends to the value 3 as t tends to infinity; this holds whatever the initial conditions.

5.4 Particular solutions

We consider second-order equations

$$\frac{d^2x}{dt^2}+q\frac{dx}{dt}+px=h$$

where p, q are real numbers and h is a function $t\mapsto h(t)$. The aim is to find a particular solution.

In practice the best method is inspired guesswork. However, we give an alternative method which is surer but often slower; it can be extended to higher order equations, and to the case that p, q are non-constant functions. This is the method of *variation of parameters*.

Suppose that the solutions of the equation

$$\frac{d^2x}{dt^2}+q\,\frac{dx}{dt}+px=0$$

have been found in the form

$$x=c_1x_1+c_2x_2$$

where c_1, c_2 are arbitrary real numbers. Consider functions of the form

$$x=v_1x_1+v_2x_2$$

where v_1, v_2 are non-constant functions. Then

$$\frac{dx}{dt}=\left(\frac{dv_1}{dt}\,x_1+\frac{dv_2}{dt}\,x_2\right)+\left(v_1\,\frac{dx_1}{dt}+v_2\,\frac{dx_2}{dt}\right).$$

Suppose that

$$\frac{dv_1}{dt}\,x_1+\frac{dv_2}{dt}\,x_2=0.$$

Then we have

$$\frac{d^2x}{dt^2}=\left(\frac{dv_1}{dt}\,\frac{dx_1}{dt}+\frac{dv_2}{dt}\,\frac{dx_2}{dt}\right)+\left(v_1\,\frac{d^2x_1}{dt^2}+v_2\,\frac{d^2x_2}{dt^2}\right)$$

and hence

$$\begin{aligned}\frac{d^2x}{dt^2}+q\,\frac{dx}{dt}+px&=\left(\frac{dv_1}{dt}\,\frac{dx_1}{dt}+\frac{dv_2}{dt}\,\frac{dx_2}{dt}\right)+v_1\left(\frac{d^2x_1}{dt^2}+q\,\frac{dx_1}{dt}+px_1\right)\\&\qquad+v_2\left(\frac{d^2x_2}{dt_2}+q\,\frac{dx_2}{dt}+px_2\right)\\&=\frac{dv_1}{dt}\,\frac{dx_1}{dt}+\frac{dv_2}{dt}\,\frac{dx_2}{dt}.\end{aligned}$$

It follows that x will be a solution of

$$\frac{d^2x}{dt^2}+q\,\frac{dx}{dt}+px=h$$

if v_1, v_2 satisfy the equations

$$\frac{dv_1}{dt}\,x_1+\frac{dv_2}{dt}\,x_2=0,$$

$$\frac{dv_1}{dt}\,\frac{dx_1}{dt}+\frac{dv_2}{dt}\,\frac{dx_2}{dt}=h.$$

Elimination gives the equations

$$\frac{dv_1}{dt}\left(x_1\frac{dx_2}{dt}-x_2\frac{dx_1}{dt}\right)=-x_2h$$

and

$$\frac{dv_2}{dt}\left(x_1\frac{dx_2}{dt}-x_2\frac{dx_1}{dt}\right)=x_1h$$

which may be solved to give (dv_1/dt), (dv_2/dt); and then integrated to give functions v_1, v_2. Of course in a specific example it may not be possible to perform the integration explicitly. We give two examples in which it is possible.

(i) Consider the equation

$$\frac{d^2x}{dt^2}+x=\frac{1}{\cos t}, \qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

Every solution of $(d^2x/dt^2)+x=0$ has the form $c_1x_1+c_2x_2$ where $x_1=\cos t$ and $x_2=\sin t$. Since

$$x_1\frac{dx_2}{dt}-x_2\frac{dx_1}{dt}=(\cos t)^2+(\sin t)^2=1$$

the equations for v_1, v_2 become $(dv_1/dt)=-(\sin t/\cos t)$ and $(dv_2/dt)=1$. Hence a particular solution is

$$\begin{aligned} x &= (\log\cos t)x_1+tx_2 \\ &= (\log\cos t)\cos t+t\sin t \end{aligned}$$

and every solution has the form

$$x=c_1\cos t+c_2\sin t+(\log\cos t)\cos t+t\sin t.$$

(ii) Consider the equation

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}-2x=e^t+\sin t-4t.$$

In this case we compare the method of variation of parameters outlined above with the method of inspired guesswork.

Inspired guesswork

The equation

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}-2x=-4t+2$$

has a solution $x = 2t$. Therefore the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -4t \tag{1}$$

has a solution $x = 2t + 1$.

The equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = e^t$$

is likely to have a particular solution ce^t; substitution shows that this means $(1 + 1 - 2)c = 1$ which is impossible. As a second guess try cte^t; this time $(c + c + ct) + (c + ct) - 2ct = 1$ and therefore $c = 1/3$. Thus $x = (1/3)te^t$ is a particular solution of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = e^t. \tag{2}$$

The equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = \sin t$$

is likely to have a particular solution $a \cos t + b \sin t$; inspection shows that this means

$$(-a + b - 2a) \cos t + (-b - a - 2b) \sin t = \sin t.$$

Therefore $b = 3a$ and $a = -1/10$. Thus

$$x = -\frac{1}{10} \cos t - \frac{3}{10} \sin t$$

is a particular solution of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = \sin t. \tag{3}$$

Now using the fact that the equation is linear, we can add the particular solutions obtained for equations (1), (2), (3) and assert that the sum

$$x = 2t + 1 + \frac{1}{3} te^t - \frac{1}{10} \cos t - \frac{3}{10} \sin t$$

is a particular solution of the original equation (ii).

Variation of parameters

The equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0$$

has solutions $c_1x_1 + c_2x_2$ where $x_1 = e^{-2t}$ and $x_2 = e^t$. Now

$$x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} = e^{-2t}e^t + 2e^{-2t}e^t = 3e^{-t}$$

so that the equations for v_1, v_2 become

$$3\frac{dv_1}{dt} = -e^{2t}(e^t + \sin t - 4t) = -e^{3t} - e^{2t}\sin t + 4te^{2t},$$

$$3\frac{dv_2}{dt} = e^{-t}(e^t + \sin t - 4t) = 1 + e^{-t}\sin t - 4te^{-t},$$

and corresponding integrals are

$$3v_1 = -\frac{1}{3}e^{3t} + \frac{1}{5}e^{2t}(-2\sin t + \cos t) + 2te^{2t} - e^{2t},$$

$$3v_2 = t - \frac{1}{2}e^{-t}(\sin t + \cos t) + 4te^{-t} + 4e^{-t}.$$

Thus the particular solution obtained for the original equation (ii) is

$$\left(-\frac{1}{9}e^t + \frac{1}{15}(-2\sin t + \cos t) + \frac{2}{3}t - \frac{1}{3}\right)$$

$$+\left(\frac{1}{3}te^t - \frac{1}{6}(\sin t + \cos t) + \frac{4}{3}t + \frac{4}{3}\right)$$

$$= -\frac{1}{9}e^t + 2t + 1 + \frac{1}{3}te^t - \frac{1}{10}\cos t - \frac{3}{10}\sin t.$$

Note that the two methods yield particular solutions which differ by $(1/9)e^t$; this is a solution of $(d^2x/dt^2) + (dx/dt) - 2x = 0$ in accordance with the second lemma of 5.1.

Exercises

1 Let $T: \mathbf{V} \to \mathbf{V}$ be the function which associates to each $x \in \mathbf{V}$ the element $T(x) = y \in \mathbf{V}$ described below. In which cases is T a linear operator?

(i) $y(t) = 1$, (ii) $y(t) = 0$,

(iii) $y(t) = (x(t))^2$, (iv) $y(t) = t^2(dx/dt)$,

(v) $y(t) = \int_0^t x(u)du$, (vi) $y(t) = t$.

2 Prove that every real number c defines a linear operator which associates to each $x \in \mathbf{V}$ the element cx.

Prove that if S, T are linear operators then so are the functions ST and $S+T$ defined by

$$(ST)(x) = S(T(x)),$$
$$(S+T)(x) = S(x) + T(x).$$

Deduce that $T^r = T \dots T$ (r factors) is a linear operator, and that if a, b are real numbers then

$$(T+a)(T+b) = T^2 + (a+b)T + ab = (T+b)(T+a).$$

3 Let T be the linear operator which associates to $x \in \mathbf{V}$ the function $t \mapsto tx(t)$. Prove that $DT - TD = 1$.

4 Prove that if m is a real number $(D-m)(e^{mt}g) = e^{mt}Dg$ for any $g \in \mathbf{V}$. Deduce that $(D-m)^r(e^{mt}g) = e^{mt}D^r g$. Hence show that $x = (c_0 + c_1 t + \cdots + c_{r-1}t^{r-1})e^{mt}$ is a solution of the equation $(D-m)^r x = 0$ for any real numbers $c_1, \dots, c_n$.

5 Prove that the equation $m^3 + m^2 + m + 6$ has a root $m = -2$. Show that the equation

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \frac{dx}{dt} + 6x = 0$$

is not stable (even though the auxiliary equation has positive coefficients).

6 Use the method of variation of parameters to find a particular solution to the equation

$$\frac{d^2x}{dt^2} - x = \frac{2}{(1+e^t)^2}.$$

7 Find all solutions of the following equations; also the solution with $x = 0$, $dx/dt = 1$ when $t = 0$.

(i) $\dfrac{d^2x}{dt^2} - x = t^2$, (ii) $\dfrac{d^2x}{dt^2} - x = e^t$,

(iii) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 4x = e^t \cos 2t$, (iv) $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + x = \sin 2t$,

(v) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = t \sin t$, (vi) $\dfrac{d^2x}{dt^2} + x = \cos t$.

CHAPTER SIX

Guide to further reading

The reader interested primarily in applications of differential equations will find his future reading determined by the fields in which he works. In chemistry, economics, engineering and physics there are many textbooks dealing in greater detail with particular differential equations which arise, and with the methods needed for their solution. The present chapter is intended rather to give an indication of the further development of the mathematical theory, and to explain why anyone interested in differential equations should also be interested in linear algebra and analysis.

6.1 Linear algebra

Many of the results of Chapters 4 and 5 are special cases of theorems about vector spaces. For the present discussion it is not necessary to know the precise definition of a vector space. The essential feature is that, if v and w are two elements of a vector space $\mathbf{W}$, then so is the linear combination $av + bw$ for any real numbers a, b. Examples of vector spaces which occur in the previous chapters include:

- $\mathbf{R}$ set of all real numbers x,
- $\mathbf{R}^2$ set of all pairs $v = (x, y)$ of real numbers,
- $\mathbf{R}^n$ set of all n-ples $v = (x_1, \ldots, x_n)$ of real numbers,
- $\mathbf{V}$ set of all infinitely differentiable functions $t \mapsto x(t)$ with domain $\mathscr{D}$,
- $\mathbf{V}^2$ set of all pairs $v = (x, y)$ with $x, y \in \mathbf{V}$.

If $\mathbf{W}$, $\mathbf{W}'$ are two vector spaces, a function $\phi:\mathbf{W}\to\mathbf{W}'$ is a rule which associates an element $\phi(w)\in\mathbf{W}'$ to each $w\in\mathbf{W}$. The function $\phi:\mathbf{W}\to\mathbf{W}'$ is a *linear map* if

$$\phi(av+bw)=a\phi(v)+b\phi(w)$$

for all a, $b\in\mathbf{R}$ and v, $w\in\mathbf{W}$. For example the linear operators of 5.1 are linear maps $T:\mathbf{V}\to\mathbf{V}$. Similarly the function

$$(x, y)\mapsto(rx+sy,\ -px-qy)$$

of 4.4 is a linear map $\phi:\mathbf{R}^2\to\mathbf{R}^2$. Many theorems about differential equations are best understood in terms of results on vector spaces and linear maps. For this purpose the following concepts are important.

The *kernel* of a linear map $\phi:\mathbf{W}\to\mathbf{W}'$ is the set

$$\mathbf{W}_0=\{w\in\mathbf{W}:\phi(w)=0\}.$$

The definition of linear map implies that $\phi(v)=\phi(w)$ if and only if $v-w$ is in $\mathbf{W}_0$. The set $\mathbf{W}_0$ is itself a vector space, since if v, $w\in\mathbf{W}_0$ then $av+bw\in\mathbf{W}_0$. The lemmas in 5.1 are special cases of these properties.

The *dimension* of a vector space $\mathbf{W}$, if it exists, is the minimum integer n for which the following result holds: there exist elements $w_1, \ldots, w_n\in W$ such that every $w\in\mathbf{W}$ can be expressed uniquely in the form

$$w=c_1w_1+\cdots+c_nw_n$$

where $c_1, \ldots, c_n$ are real numbers. If no such integer exists then the dimension of $\mathbf{W}$ is said to be infinite. For example the dimension of $\mathbf{R}^2$ is 2, since any $w\in\mathbf{R}^2$ can be expressed uniquely in the form $x(1, 0)+y(0, 1)$ where x, $y\in\mathbf{R}$; similarly the dimension of $\mathbf{R}^n$ is n. On the other hand the dimension of $\mathbf{V}$ is infinite. The main theorem of 5.2 states that, if $T:\mathbf{V}\to\mathbf{V}$ is the linear map

$$D^n+a_1D^{n-1}+\cdots+a_n$$

where $a_1, \ldots, a_n\in\mathbf{R}$, then the kernel of T has dimension n. The concept of dimension is needed to make precise intuitive ideas about 'arbitrary constants' and 'degrees of freedom'.

The *eigenvectors* of a linear map $\phi:\mathbf{W}\to\mathbf{W}$ are those elements $w\in\mathbf{W}$ such that $mw=\phi(w)$ for some real number m. We give two examples to illustrate the fact that many properties of ϕ may be

deduced from corresponding properties of the eigenvectors of ϕ. Firstly, let $D:\mathbf{V}\to\mathbf{V}$ be the linear map $x\mapsto dx/dt$ of 5.1. Its eigenvectors are the functions $x=ce^{mt}$ where c, $m\in\mathbf{R}$. Secondly, let $\phi:\mathbf{R}^2\to\mathbf{R}^2$ be the linear map $(x, y)\mapsto(rx+sy, \ -px-qy)$ of 4.4 Its eigenvectors are the elements $(x_1, y_1)\in\mathbf{R}^2$, such that

$$mx_1 = rx_1 + sy_1$$

$$my_1 = -px_1 - qy_1$$

for some $m\in\mathbf{R}$, which played a central role in the proof of 4.5.

The basic properties of vector spaces, linear maps, kernels, dimension and eigenvectors may be found in many textbooks; for example

K. Hoffmann and R. Kunze, *Linear Algebra* (Prentice-Hall, New Jersey, 1961),

N. H. Kuiper, *Linear Algebra and Geometry* (North-Holland, Amsterdam, 1962).

6.2 Analysis

It has been mentioned in 1.7 that a precise definition of differentiation and derivative requires the concept of a continuous function $f:\mathscr{D}\to\mathbf{R}$. Here the domain $\mathscr{D}$ is contained in $\mathbf{R}$, and f is a rule which associates a real number $f(t)$ to each $t\in\mathscr{D}$. If $\mathscr{D}$ is an interval then every differentiable function is continuous, and every continuous function is integrable. However, the converse statements are false. All properties of differentiable functions, including assumptions 1.10 and 1.12, the chain rule, the rules for differentiating sums and products, and the precise definitions of trigonometric and exponential functions, depend ultimately on the properties of continuous functions developed in a first-year university analysis course. Suitable references are

R. M. F. Moss and G. T. Roberts, *A Preliminary Course in Analysis* (Chapman and Hall, London, 1968),

T. M. Flett, *Mathematical Analysis* (McGraw-Hill, London, 1966).

The latter book discusses in addition the continuity and differentiability of functions $f:\mathscr{D}\to\mathbf{R}^n$ where the domain $\mathscr{D}$ is contained in $\mathbf{R}^m$. This leads to the definition of partial derivatives, and to such

results as the chain rule and inverse function theorem for functions of several variables. The study of such functions leads to partial differential equations; these have a vast literature and many applications, but a discussion of these would lead far away from the scope of this book. We mention, however, three examples in which functions of this type have already occurred in earlier chapters.

(i) The differential equation

$$\frac{dx}{dt} = f(t, x), \qquad (t, x) \in \mathscr{S},$$

considered in 3.5 involves a function $f: \mathscr{S} \to \mathbf{R}$ where $\mathscr{S}$ is a region in $\mathbf{R}^2$. This is the case $m = 2$, $n = 1$.

(ii) The linear system

$$\frac{dx}{dt} = rx + sy$$

$$\frac{dy}{dt} = -px - qy$$

considered in 4.4 can be written in the form

$$\frac{dv}{dt} = \phi(v), \qquad v = (x, y) \in \mathbf{R}^2,$$

where $\phi: \mathbf{R}^2 \to \mathbf{R}^2$ is the linear map mentioned at the end of 6.1. Solutions are functions $v: \mathbf{R} \to \mathbf{R}^2$ which associate to each $t \in \mathbf{R}$ the point $v(t) = (x(t), y(t)) \in \mathbf{R}^2$.

(iii) The differential equation

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_n x = 0$$

can be turned into a system of first-order equations by substituting $x_1 = x$, $x_2 = dx/dt, \ldots, x_n = d^{n-1}x/dt^{n-1}$. It can then be written

$$\frac{dx_1}{dt} = x_2$$

$$\cdots$$

$$\frac{dx_{n-1}}{dt} = x_n$$

$$\frac{dx_n}{dt} = -a_1 x_n - \cdots - a_n x_1.$$

If $\phi:\mathbf{R}^n \to \mathbf{R}^n$ is the linear map which associates to

$$v=(x_1, \ldots, x_n) \in \mathbf{R}^n$$

the element

$$\phi(v)=(x_2, \ldots, x_n, \ -a_1x_n - \cdots - a_nx_1)$$

of $\mathbf{R}^n$, the equation $T(x)=0$ of 5.2 is equivalent to the equation

$$\frac{dv}{dt}=\phi(v), \qquad v=(x_1, \ldots, x_n) \in \mathbf{R}^n.$$

Solutions are functions $v:\mathbf{R} \to \mathbf{R}^n$ which associate to each $t \in \mathbf{R}$ the point $v(t)=(x_1(t), \ldots, x_n(t)) \in \mathbf{R}^n$.

6.3 Existence theorems

The main question left open in Chapter 3 is: for what regions $\mathscr{S}$ in $\mathbf{R}^2$, and for what functions $f:\mathscr{S} \to \mathbf{R}$, does the equation

$$\frac{dx}{dt}=f(t, x), \qquad (t, x) \in \mathscr{S},$$

have a solution through every point $(t_0, x_0) \in \mathscr{S}$, and when is this solution unique? Theorems 3.1 and 3.3 answer the question in the special cases $f(t, x)=g(t)$ and $f(t, x)=h(x)$. What is required is a theorem which does not give such complete information in these special cases, but which on the other hand applies to a more general class of functions f. For simplicity we assume that $\mathscr{S}$ is a rectangle $\{(t, x) \in \mathbf{R}^2 : p < t < q \text{ and } a < x < b\}$ in $\mathbf{R}^2$.

Existence theorem. *Let $f:\mathscr{S} \to \mathbf{R}$ be a continuous function, and suppose $(t_0, x_0) \in \mathscr{S}$. There is a real number $h>0$ such that the equation*

$$\frac{dx}{dt}=f(t, x)$$

has a solution $t \mapsto x(t)$ with domain $\mathscr{D}=\{t \in \mathbf{R} : t_0-h < t < t_0+h\}$ and $x(t_0)=x_0$.

Uniqueness theorem. *Let $f:\mathscr{S} \to \mathbf{R}$ be a continuous function for which the partial derivative $\partial f/\partial x$ is also continuous, and suppose $(t_0, x_0) \in \mathscr{S}$. If the equation*

$$\frac{dx}{dt}=f(t, x)$$

has two solutions x *and* x *with* $x(t_0)=\tilde{x}(t_0)=x_0$ *and domain* $\mathscr{D}=\{t\in\mathbf{R}: t_0-h<t<t_0+h\}$ *then* $x(t)=\tilde{x}(t)$ *for all* $t\in\mathscr{D}$.

These two theorems imply immediately the truth of assumptions 1.10 and 1.12. The necessity of the additional hypothesis (continuity of the partial derivative $\partial f/\partial x$) in the uniqueness theorem is shown by examples 2.5 and 3.4.

Both theorems have immediate extensions to the case of a differential equation

$$\frac{dv}{dt}=f(t, v)$$

where $v=(x_1, \ldots, x_n)\in\mathbf{R}^n$ and $(t, v)=(t, x_1, \ldots, x_n)$ lies in a region $\mathscr{S}$ of $\mathbf{R}^{n+1}$. It is this extension, applied to example (ii) of 6.2, which shows that the linear system of 4.4 has only those solutions found in 4.5. The same extension, applied to example (iii) of 6.2 shows that every solution of the linear equation with constant coefficients $T(x)=0$ is of the form claimed in 5.2.

Other extensions of the existence and uniqueness theorems deal with the question whether the solution through (t_0, x_0) depends continuously on (t_0, x_0), and with the properties of approximate solutions. Here the theoretical discussion is also of great practical importance, because it is often possible to compute a numerical solution by obtaining closer and closer approximations to it. All these matters are discussed in more advanced textbooks on ordinary differential equations, such as

E. A. Coddington and N. Levinson, *Theory of ordinary differential equations* (McGraw-Hill, New York, 1955),

S. Lefschetz, *Differential equations: geometric theory* (Interscience, New York; 2nd edition 1962),

W. Hurewicz, *Lectures on ordinary differential equations* (M.I.T. Press, Cambridge, Mass., 1958).

The latter book is also the best introduction to the qualitative theory of differential equations, in which one adopts the approach of 4.5 and asks (in higher dimensions, or for a non-linear differential equation) what patterns of curves can arise, and which equations are structurally stable.

Notes on exercises

Introduction

1. y can have arbitrarily small values when $x < 0$. **2.** A partially correct statement is $y = \sin(a + x)$ for $-\frac{1}{2}\pi \leqslant a + x \leqslant \frac{1}{2}\pi$ (see 3.4 for a complete statement). **3.** Function $u = 1/x$ is not defined when $x = 0$. **4.** First function never makes sense, second function only when $x = 1$. **5.** Consider regions $x > 0$ and $x < 0$ separately. **6.** Solution has distinct formulae for $x > 0$ and $x < 0$.

Chapter One

1. Domains of functions given are $x \geqslant 0$, $x > 0$, all x, $x > 0$, $x > 1$, $2n\pi < x < (2n+1)\pi$, empty set. **2.** (i) $a + 2\sqrt{x}$ for $x > 0$. (ii) $a + \log x$ for $x > 0$, and $b + \log(-x)$ for $x < 0$. (iii) $a + \sin x$ for all x. (iv) $a_n + \log \sin x$ for $2n\pi < x < (2n+1)\pi$, and $b_n + \log(-\sin x)$ for $(2n-1)\pi < x < 2n\pi$. (v) $a + \log \log x$ for $x > 1$ and $b + \log(-\log x)$ for $0 < x < 1$. (vi) $a + \log(-\log \sin x)$ for $2n\pi < x < (2n+1)\pi$.

Chapter Two

1. First sketch slope lines along $x = 0$, $t = 0$, $x = \pm t$, then fill in solution curves. **2.** Equations are defined in regions $t \geqslant 0$, all t, $t \geqslant 0$, $t \geqslant 0$, all t, but first two equations are of second degree with two solution curves through each point (t, x) with $t > 0$. **3.** If $t \neq 0$ slopes at (t, x) are $(1/2t)[x \pm \sqrt{(x^2 - 4t)}]$. Hence no solutions through (t, x) if $x^2 < 4t$, two solutions if $x^2 > 4t$. If $x^2 = 4t$ slope

is $x/2t = 2/x$, that is tangent to parabola $x^2 = 4t$. Therefore $x^2 = 4t$ is a solution curve. Differentiation of equation gives

$$\left(\frac{dx}{dt}\right)^2 + 2t\,\frac{dx}{dt}\,\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - x\,\frac{d^2x}{dt^2} = 0.$$

Therefore either $2t(dx/dt) = x$ (in which case $x^2 = 4t$) or $d^2x/dt^2 = 0$ (in which case solution curves are straight lines). Therefore all other solution curves are tangents to the parabola.

Chapter Three

1. (i) $x = a + t^3 + 2t^2$ with $a = -3$ for solution through (1, 0). (ii) $x = a + be^t$ with $a = -be$. (iii) $x = a + \tan^{-1} t$ with $a = -\frac{1}{4}\pi$. (iv) $x = a + \sinh^{-1} t$ with $a = -\log(1+\sqrt{2})$. (v) $x = a + \sin t$ with $a = -\sin 1$. (vi) $x = a_n + \log \sin t$ for $2n\pi < t < (2n+1)\pi$, and $x = b_n + \log(-\sin t)$ for $(2n-1)\pi < t < 2n\pi$ with $a_0 = -\log \sin 1$, b_n arbitrary. **2.** (i) $x = 1$ or $x = 2$ or $x = (ae^t - 2)/(ae^t - 1)$ where $a \neq 0$ with domain $\mathbf{R}$. (ii) $x = -\log(a - bt)$ with domain $\{t \in \mathbf{R}: a > bt\}$; choose $a = e^{-1}$ to get unique solution through (0, 1). (iii) $x = 1$ or $x = 1 - (1/t - c)$ with domain $t \neq c$. (iv) $x = 1$ or $x = \cosh(t - c)$ with domain $t \geqslant c$; solution through any point on $x = 1$ is not unique. (v) $x = 0$ or $x = (t - c)^2$ with domain $t \geqslant c$; choose $c = -1$ to get unique solution through (0, 1). (vi) $x = 0$ or $x = \sin^{-1}(ae^t)$ where $a \neq 0$ with domain $t \leqslant -\log|a|$; choose $a = \sin 1$ to get unique solution through (0, 1). **3.** (i) Either $t = 0$ or $x = 0$ or

$$\frac{1 - 3t}{t^2} = (2x + x^3)\,\frac{dx}{dt}$$

which has variables separate. (ii) Since $t > 2$ either $x = -\frac{1}{2}$ or

$$\frac{1}{t^2 - 4} = \left(\frac{1}{1 + 2x}\right)\frac{dx}{dt}.$$

(iii) Substitute $u = x/t$. (iv) Since $t > x > 0$ equation can be written

$$t\,\frac{du}{dt} = \frac{u^3}{1 - u^2}$$

where $u = x/t$. (v) Equation becomes $(d/dt)(xe^{3t}) = 2t$. (vi) Equation becomes $(d/dt)(t^2 + 3tx - \frac{1}{2}x^2) = t^2$. **4.** (i) Write in form

$$\frac{d}{dt}\,(xe^{2t}) = e^{3t}.$$

(ii) $(d/dt)(x/\cos t) = 0$. (iii) $(d/dt)(x \cos t) = 2 \sin 2t$. (iv)

$$\frac{d}{dt}\left(\frac{x}{t^3} e^{-1/t^2}\right) = \frac{1}{t^3} e^{-1/t^2}.$$

(v) Substitute $y = x^{-3}$. (vi) Substitute $u = \log t$. **5.** If $t = -e^u$ then

$$\frac{dx}{du} = \frac{dt}{du}\frac{dx}{dt} = t\frac{dx}{dt}.$$

6. Existence of derivatives at $t = 0$ implies (i) no restrictions, (ii) $p = r$, (iii) $p = r$ and $q = s$.

Chapter Four

1. $x = x_0 e^{-\lambda t}$ has $x = \frac{1}{2}x_0$ when $-\lambda t = \log \frac{1}{2} = -\log 2$. **2.** $(0, e^{2t})$ is a solution along $x = 0$, and $(e^t, -e^t)$ is a solution along $x + y = 0$. Hence $(ae^t, be^{2t} - ae^t)$ is a solution. Choose $a = x_0$, $b = x_0 + y_0$. **3.** Substitution $t = e^u$ gives

$$\frac{d^2x}{du^2} - 3\frac{dx}{du} + 2x = 0$$

and hence $x = ae^u + be^{2u} = at + bt^2$ for $t > 0$. In region $t < 0$ substitute $t = -e^u$. **4.** (i) $(d^2x/dt^2) - 2(dx/dt) + 4x = 0$. (ii) $(d^2x/dt^2) + (dx/dt) - 2x = 0$. (iii) $(d^2x/dt^2) + 2(dx/dt) + x = 0$. **5.** (i) $x = ae^{2t} + be^{3t}$. (ii) $x = (a + bt)e^{2t}$. (iii) $x = e^{-2t}(a \cos t + b \sin t)$. (iv) $x = a + be^{-3t}$. **6.** (i) $x = -e^t + e^{2t}$. (ii) $x = \sin t$. (iii) $x = te^{-t}$. (iv) $x = e^t \sin t$.

Chapter Five

1. (i) (iii) (vi) non-linear, (ii) (iv) (v) linear.

2.
$$\begin{aligned}(ST)(c_1x_1 + c_2x_2) &= S(T(c_1x_1 + c_2x_2))\\ &= S(c_1T(x_1) + c_2T(x_2))\\ &= c_1ST(x_1) + c_2ST(x_2);\end{aligned}$$

$S + T$ similar;

$$\begin{aligned}(T+a)(T+b)x &= (T+a)(Tx+bx)\\ &= (T+a)Tx + b(T+a)x\\ &= (T^2+aT+bT+ab)x.\end{aligned}$$

3. $DT(x) = D(tx) = x + tD(x) = x + TD(x)$.

4. $D(e^{mt}g) = D(e^{mt})g + e^{mt}D(g) = e^{mt}mg + e^{mt}D(g)$. If g is a solution $(c_0 + c_1t + \cdots + c_{r-1}t^{r-1})$ of $D^r(g) = 0$ then $x = e^{mt}g$ is a solution of $(D-m)^r(x) = 0$.

5. $m^3 + m^2 + m + 6 = (m+2)(m^2 - m + 3)$ and roots of $m^2 - m + 3$ have positive real part.

6. Choose $x_1 = e^{-t}$ and $x_2 = e^t$. Equations for v_1, v_2 become

$$\frac{dv_1}{dt} = \frac{-e^t}{(1+e^t)^2}, \qquad \frac{dv_2}{dt} = \frac{e^{-t}}{(1+e^t)^2}.$$

Integration gives $v_1 = 1/(1+e^t)$. The substitution $u = e^{-t}$ yields $v_2 = 2\log(1+e^{-t}) + 1/(1+e^{-t}) - e^{-t}$. Therefore there is a particular solution:

$$x = 2e^t \log(1+e^{-t}) + \frac{e^t}{1+e^{-t}} + \frac{e^{-t}}{1+e^t} - 1.$$

7. (i) $x = ae^t + be^{-t} - t^2 - 2$ where a, b are arbitrary real numbers; for solution such that $x = 0$, $dx/dt = 1$ when $t = 0$ choose $a = 3/2$, $b = 1/2$. (ii) $x = ae^t + be^{-t} + \frac{1}{2}te^t$ with $a = 1/4$, $b = -1/4$. (iii) $x = ae^{-t}\cos(t\sqrt{3}) + be^{-t}\sin(t\sqrt{3}) + (8/73)e^t\sin 2t + (3/73)e^t\cos 2t$ with $a = -3/73$, $b = 17\sqrt{3}/73$. (iv)

$$x = ae^{\frac{1}{2}t}\sin\left(t\,\frac{\sqrt{3}}{2}\right) + be^{\frac{1}{2}t}\cos\left(t\,\frac{\sqrt{3}}{2}\right) + \frac{2}{13}\cos 2t - \frac{3}{13}\sin 2t$$

with $a = 40/13\sqrt{3}$, $b = -2/13$. (v)

$$x = ae^{-t} + be^{-3t} + \frac{1}{10}(t\sin t - 2t\cos t) + \frac{2}{50}\sin t + \frac{11}{50}\cos t$$

with $a = 1/4$, $b = -47/100$. (vi) $x = a\cos t + b\sin t + \frac{1}{2}t\sin t$ with $a = 0$, $b = 1$.

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